The Quantum Aether Model: Rebuilding Physics from First Principles

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Part I

Spectral Cosmology: Origins and Evolution

Chapter 1

Foundations: Principles, States, and the Quiescent Void

1.1 A Spectral Manifesto: Rewriting Cosmic Beginnings

Thesis. The universe does not *begin* on a pre-given stage; rather, the stage itself—effective spacetime, its clocks, its rulers, and its "laws"—*emerges* from the internal dynamics of a pre-geometric medium: the Quantum–Aether. This volume advances a constructive alternative to the standard Big-Bang narrative by deriving inflation-like expansion, the seeding of structure, and late-time acceleration from spectral dynamics of the aether itself, with no ad hoc inflaton and no primordial singularity.

From postulates to production. Volume I established the five spectral fields as ontological primitives,

$$S(x) \equiv \{\Phi,\ \rho,\ \eta,\ \chi^{\mu},\ T\},$$

the spectral derivative D_μ and the emergent connection ω_μ, and showed how an effective Lorentzian metric $g^{\text{eff}}_{\mu\nu}[S]$ arises from correlations of these fields. Here we turn those ingredients into a cosmological engine. Our first principles are operational and testable:

1. **No background.** No geometry or probabilistic structure is imposed *a priori*. All observables must be built from S and its spectral derivatives (cf. Vol. I, Ch. 2–3).

2. **No singularity.** Divergences signal breakdown of an *effective* description, not infinities in the medium. Ordered aetheric states admit finite action density and regular g^{eff}.

3. **No event before time.** "Events" presuppose temporal ordering. Before an internal time order is activated (via T and dissipative channels controlled by $\eta > 0$), "before/after" has no physical meaning.

Why a new beginning? A critique of the standard script. The ΛCDM + inflaton framework has unmatched empirical reach as a *fitting* scheme, but it remains agnostic about origins and dependent on *assumptions*:

- *Singularity and initial data.* The classical Big Bang is a geodesic incompleteness with indeterminate initial conditions; the measure problem in inflation leaves the "initial patch" unexplained.

- *Inflaton sector.* Potentials and couplings are *posited* (plateau, hilltop, ...); trans-Planckian concerns and eternal-inflation pathologies linger.

- *Cosmological constant.* A rigid Λ is inserted with extreme fine-tuning, without dynamical origin or environmental correlation.

Our response is *medium first*: inflationary expansion, perturbation spectra, and late-time acceleration appear as *responses* of a dynamical substrate, not as extra ingredients bolted onto spacetime.

From quiescence to law: birth and evolution of physical *rules*

We articulate a program for the *origin and development of laws* that closes the usual metaphysical gap.

(1) Before laws: a static generative structure. Conceptual point zero is a homogeneous, ordered aetheric state S_0 with constant values of $\{\rho, \eta, \chi^\mu, T, \Phi\}$ and a stationary unified spectral action $\mathcal{A}_{\text{spec}}[S]$ (Vol. I, Ch. 3). There is no operational time yet; T has not activated clock-like order. What exists is a *generative code*—the spectral action—specifying the *space of possibilities*, not a time-evolution law.

(2) Birth of laws: quiescence loss ⇒ time order ⇒ effective rules. A modulational instability or threshold crossing in field space (not "in time") opens dynamical channels. The variable T becomes a valid internal order parameter and renders *before/after* meaningful; dissipation via $\eta > 0$ selects an *arrow* by enforcing nonnegative entropy production,

$$\partial_\mu(s\, u^\mu) = \sigma_{\text{diss}} = 2\,\eta\, g^{\mu\nu}_{\text{eff}} (\nabla_\mu \Phi)^* (\nabla_\nu \Phi) \ \geq 0. \tag{1.1}$$

At that moment the generative code $\mathcal{A}_{\text{spec}}$ *appears* operationally as dynamical field equations and conservation identities.

(3) How the five fields *make* law.

- T: creates *temporal ordering*—without T there is no dynamical law at all.
- η: fixes the *direction* (irreversibility) by opening dissipative channels and defining Lyapunov functionals.
- χ^μ: organizes *compensating responses* (emergent gauge behavior and Ward-like identities) via fiber isometries.
- ρ: sets *scales and cadences* (spectral inertia, effective Planck mass $M_{\text{eff}}(\rho)$, local rates).
- Φ: provides *phase coherence* underpinning linear wave dynamics and the Hilbert-space limit at low energy.

(4) Why laws *change* (without arbitrariness). A *law* in our sense is a stabilized pair (Form, Parameters). The *Form* is imposed by symmetries/constraints of the internal fiber and director; the *Parameters run* with scale and context (spectral RG):

1. *Scale flow:* same Form, couplings $g_i(k)$ run with scale.
2. *Epochal changes:* across phases, the Form itself changes (e.g. critical unification where "forces" cease to be distinct responses).
3. *Contextual modulation:* weak space–time dependence consistent with the gates and observational bounds.

Restoration principles ensure that higher-phase laws reduce to lower-energy ones in appropriate limits.

(5) How the attendant mathematics emerges.

- *Differential geometry* arises once T and χ^μ stabilize wave propagation, producing an effective metric/connection (g^{eff}, ω).
- *Hilbert space* emerges from quadratic expansions around coherent states: a positive quadratic energy defines an inner product and linearized evolution.
- *Probability* (Bornean assignment) appears as a *constraint* in the stabilized linear regime, not a primitive axiom.
- *Gauge structure* follows from fiber isometries; Noether/Ward identities are identities of the spectral bundle, not stipulations.

(6) The arrow of *law* evolution. With T active and $\eta > 0$, theory space exhibits *attractors* (universality classes). Early/high-energy epochs favor critical phases (washed distinctions, unified responses); late/low-energy epochs settle into legally stable phases (scale separation, topological protection, quasi-constants). Thus "constants" measured today are effective parameters stabilized in our cosmic phase.

(7) Common objections & concise replies.

- *"Do laws then change capriciously?"* No. Form is fixed by symmetry in a given phase; parameters run under strict RG-like flow and observational gates.
- *"Where are the true constants?"* Metaphysical constants are replaced by *effective* constants stabilized in phase; these are what experiments access.
- *"How do phases avoid contradiction?"* By restoration: each new phase reproduces the lower-energy limit (recoverability checks are built in).

From inflation to acceleration: replacing add-ons with responses

Inflation without an inflaton. A coherent large-amplitude mode of spectral viscosity (η-wave) plays the role of an inflating driver: its coarse-grained stress yields $w_\eta \simeq -1$ while coherence lasts, with graceful exit when damping $\Gamma_\eta(\mathcal{S}) > 0$ transfers energy into Φ and structural sectors. This trades an *assumed* inflaton potential for a *derived* response of the medium.

Seeding structure without mystery. Symmetry-breaking across the instability forms defects and solitons governed by homotopy of admissible order parameters (domain walls, strings, monopoles, Hopf/Skyrme textures). Their densities follow a Kibble-like estimate from the growth-rate spectrum of linear response. These stable, non-luminous, cold configurations in ρ (and composites with χ^μ) are natural dark-matter candidates.

Cosmic web as resonance, not mere collapse. Interference of the most unstable modes builds a Chladni-like pattern whose ridges/nodes seed filaments and voids. Large-scale morphology encodes spectral anisotropies and curvature corrections, predicting statistics beyond purely gravitational collapse.

Late-time acceleration as η-relaxation. Residual ultra-long relaxation of η supplies a slowly varying effective dark-energy component with $w_{\text{eff}}(t) = -1 + \delta(t)$, naturally allowing mild time dependence and environmental correlations with the ρ-field distribution—in contrast with a rigid Λ.

Falsifiable discriminants

The manifesto stands or falls by *distinctive* signatures:

- Angle-dependent sound speed $c_s(\theta)$ and constrained $c_T \simeq 1 + \delta c_T(\mathcal{S})$ during the structural epoch.
- Non-Gaussian shapes tied to η-mode mixing and director anisotropy (departures from single-field inflaton templates).
- Defect/soliton cross-correlations (CMB lensing $\times$ LSS) consistent with stability gates.
- Late-time correlations between lensing maps and deviations $\delta(t)$ in w_{eff}.

Boxed Essay: Two Questions on Emergent Laws

Scope. This essay addresses two foundational questions that recur across Volume II: (i) what role does "infinity" play in an emergent framework? and (ii) why does symmetry appear so forcefully in nature, and how can it emerge? Links to technical material: consistency gates (§12.1), ΛCDM comparison (§12.2), statistical anisotropy and non-Gaussian shapes (§4.3), and

multi–messenger bounds on c_T (§14.3).

(1) If laws are emergent, where does "infinity" come from?

Core idea. "Infinity" is not a physical entity but a *limiting tool* that compresses potentially unbounded procedures (refinement, coarse–graining, summation). In an emergent setting it appears operationally in three ways:

- **Metric/topological completion.** Reals from Cauchy completion; one–point compactification for non–compact spaces. Infinity is a *limit of a generative process.*
- **Regulate–then–renormalize.** Define observables with a physical cutoff Λ (e.g. lattice spacing, correlation length ξ), then remove the regulator by Wilsonian flow toward a fixed point:
$$\frac{\mathrm{d}g_a}{\mathrm{d}\ln\Lambda} = \beta_a(\{g\}),$$
$$\beta_a(g^*) = 0 \;\Rightarrow\; g_a(\Lambda) \to g_a^*.$$
so that long–distance predictions become regulator–independent.
- **Failure flag for an EFT.** Genuine divergences/singularities signal the breakdown of the effective description, calling for new degrees of freedom or physical regulators (small dissipation $\eta > 0$, finite ξ, protected topological sectors).

Spectral viewpoint. In the Quantum–Aether model the pair ($\eta > 0,\ \xi < \infty$) provides *native* IR/UV control: η enforces entropy production and damps ultraviolet recurrences; ξ limits long–range correlations. "Infinity" is used as a formal limit only when the spectral RG flows to a stable law (Part VI), never as a claim of physically unbounded magnitudes.

Actionable rule. Use limiting procedures only when (i) RG flows to a fixed point, (ii) front velocities respect the emergent cone ($v_{\text{front}} \le c_{\text{eff}}$), and (iii) positivity/dispersion constraints hold; otherwise treat divergences as exit signs of the current EFT.

(2) Why does symmetry appear so strongly, and how does it emerge?

Unifying picture. Symmetry is stability under transformations. In emergent media it is *amplified* by four mechanisms:

1. **RG attractors.** Under coarse–graining, irrelevant asymmetries flow to zero; effective symmetries enlarge in the IR. For anisotropic sound, a minimal flow reads
$$\frac{d}{d\ln\Lambda}\left(c_\parallel^2 - c_\perp^2\right) = -\gamma\left(c_\parallel^2 - c_\perp^2\right) + \cdots, \qquad \gamma > 0,$$
implying $c_\parallel - c_\perp \to 0$ at large scales.
2. **Descriptive economy / maximum entropy.** Highly symmetric macrostates minimize descriptive complexity under energy/information constraints (e.g. harmonic solutions, Young–Laplace curvature).
3. **Gauge as redundancy.** Collective constraints generate representational redundancy recast as an *emergent gauge symmetry* (Vol. I, Ch. 8; Vol. II, Part VII).
4. **Spontaneous symmetry breaking (SSB).** Laws can be symmetric while selected states are not, producing ordered structures and topological defects.

Consistency gates (operational). Along spectral flows, enforce

$$\begin{aligned}
\textbf{(S) No ghosts:}\quad & \mathsf{K} \succ 0,\\
\textbf{(G) Gradient stability:}\quad & c_s^2 > 0,\ c_T^2 > 0,\\
\textbf{(C) Causality:}\quad & v_{\rm front} \le c_{\rm eff},\\
\textbf{(P) Forward positivity:}\quad & \text{dispersion/positivity for higher–derivative operators},\\
\textbf{(NS) No strong coupling:}\quad & \Lambda_{\rm sc} > \max(H,\ E_{\rm proc}).
\end{aligned}$$

These gates *favor* more symmetric effective laws in the IR, and connect directly to data via g_*, anisotropic $f_{\rm NL}$ templates (§4.3), and c_T bounds (§14.3).

Spectral ties to the model. At observable scales $\ell \gg \xi$, the emergent metric $g_{\rm eff}$ becomes more isotropic; constraints on statistical anisotropy (CMB) and on $c_T - 1$ (multi–messenger) translate into bounds on the small difference $c_\parallel - c_\perp$ and on curvature–sector couplings, consistent with the global gates (§12.1) and the ΛCDM comparison (§12.2).

One-line takeaway. Infinity is a *limit tool* justified by stable RG flow; symmetry is a *dynamical/informational attractor* of the aether, strengthened by RG, redundancy (gauge), and SSB—while the consistency gates ensure that only causal, stable, positive–energy effective laws survive to the infrared.

One-line summary

> *Laws are not imposed upon being; they are stabilized regularities emerging from the aether's dynamics after quiescence breaks: their **form** is fixed by internal symmetries, their **parameters** flow with scale and context, and under the arrow set by $\eta > 0$ they evolve toward legally stable universality classes.*

The remainder of this chapter formalizes the quiescent state S_0, maps the Volume I consistency gates to cosmological initialization, and prepares the linear-stability analysis whereby the clock T and the arrow η become operational. Subsequent chapters implement the η-wave inflationary mechanism, the statistics of defect formation, the resonance picture of the cosmic web, and the late-time η-driven acceleration, confronting each step with observational discriminants.

1.2 The Quiescent Void: Definition, Symmetries, and Linear–Perturbation Setup

Definition (homogeneous, ordered pre–geometric state)

We posit a spatially homogeneous, highly ordered reference configuration of the Quantum–Aether,

$$S_0 \;=\; \{\ \rho_0,\ \eta_0 > 0,\ \Phi_0 = 0,\ \chi^{(0)\mu},\ T_0\ \},\qquad D_\mu S_0 = 0,\qquad \partial_\mu S_0 = 0, \tag{1.2}$$

with vanishing large–scale spectral gradients and no *imposed* spacetime background. From S_0 one constructs the emergent coframe $e^a{}_\mu(S_0)$ and the effective metric

$$g^{(0)}_{\mu\nu} \;=\; g_{\rm eff}[S_0],\qquad u^\mu \text{ unit timelike},\quad u^\mu u_\mu = -1,\qquad \hat\chi^\mu \text{ unit spacelike},\quad \hat\chi^\mu\hat\chi_\mu = +1,\quad u\cdot\hat\chi = 0. \tag{1.3}$$

Here u^μ is the aether rest frame selected by S_0 (Vol. I, Ch. 2), and $\hat\chi^\mu$ is the normalized director built from $\chi^{(0)\mu}$. The internal time T is *affine* in this state (constant gradient in the u^μ frame), but does not yet define a cosmic history until dynamics is switched on by instabilities (Sec. **??**).

Background connection. We adopt the Palatini split (Vol. I, Ch. 2–3)

$$\omega_\mu{}^{ab} \;=\; \omega_\mu{}^{ab}[e] \;+\; K_\mu{}^{ab}(S),\qquad K_\mu{}^{ab}(S_0)\ \text{ constant and antisymmetric in } (ab), \tag{1.4}$$

so that the background curvature $R(\omega, e)$ is homogeneous, with $F_{\rm spec}(S_0) = 0$ in the strictly quiescent limit. Metric–compatibility $\nabla^\omega g^{(0)} = 0$ holds by construction.

Symmetries and gates at S_0

The quiescent void exhibits:

- *Spatial homogeneity* and a residual *anisotropy* encoded by $\hat{\chi}^\mu$ (an internal alignment that may softly break full isotropy without violating observational bounds once dynamics begins).
- A global $U(1)_\Phi$ phase symmetry with $\Phi_0 = 0$; internal fiber isometries of F act trivially on S_0.
- **Consistency gates** (Vol. I): (i) no–ghost: positive time–kinetic forms for small fluctuations; (ii) causal hyperbolicity: principal cones inside those of $g^{(0)}$; (iii) nonnegative entropy production for $\eta_0 > 0$ once dynamics turns on.

Coarse–graining and emergent kinematics

Cosmological statements are made after coarse–graining on a comoving scale L:

$$\langle \mathcal{O} \rangle_L(x) \;=\; \int d^4y\, W_L(x-y)\, \mathcal{O}(y),$$

with a normalized, compactly supported kernel W_L. The expansion of the aether frame defines an emergent Hubble rate

$$H \;\equiv\; \frac{1}{3}\nabla_\mu u^\mu, \qquad \gamma_{\mu\nu} \;\equiv\; g^{(0)}_{\mu\nu} + u_\mu u_\nu \quad \text{(spatial metric in the } u^\mu \text{ frame).} \tag{1.5}$$

In the exactly quiescent state $H = 0$; a nonzero H appears only once coherent dynamics is excited (Sec. **??**).

Linear perturbations around S_0

Let $\delta\mathbf{S}$ collect the independent fluctuations,

$$\delta\mathbf{S} \;\equiv\; \big(\delta\rho,\ \delta\eta,\ \delta\alpha,\ \delta\chi_I,\ \delta\Phi,\ \delta e^a{}_\mu,\ \dots\big)^T, \qquad \alpha \equiv \arg\Phi. \tag{1.6}$$

In *comoving–aether gauge* we set $\delta T = 0$ and work in the u^μ frame; tensor fluctuations will later be put in TT gauge. The quadratic Lagrangian density takes the canonical form

$$\mathcal{L}^{(2)} \;=\; \tfrac{1}{2}\,\delta\mathbf{S}^T\left[\mathsf{K}\,\partial_t^2 \;+\; \mathsf{C}\,\partial_t \;+\; \mathsf{Z}^{ij}\,\nabla_i\nabla_j \;+\; \mathsf{M}^2\right]\delta\mathbf{S}, \tag{1.7}$$

where all coefficient operators are homogeneous tensors constructed from S_0, the spectral derivative D_μ, and ω_μ.

No–ghost and dissipation gates. The *time–kinetic* block must be positive definite,

$$\mathsf{K} \;\succ\; 0, \tag{1.8}$$

in every propagating sector. Spectral viscosity $\eta_0 > 0$ induces an anti–Hermitian part in C (Rayleigh–type damping). The entropy production density is nonnegative (Vol. I),

$$\partial_\mu(s\,u^\mu) \;=\; \sigma_{\text{diss}} \;=\; 2\,\eta_0\, g^{(0)\mu\nu}\,(\nabla_\mu\Phi)^*(\nabla_\nu\Phi) \;\geq 0, \tag{1.9}$$

ensuring the arrow–of–time gate once dynamics is activated.

Anisotropic decomposition and projectors

Define the spatial projectors (in the u^μ frame)

$$\Pi^{ij}_\parallel \;\equiv\; \hat{\chi}^i\hat{\chi}^j, \qquad \Pi^{ij}_\perp \;\equiv\; \gamma^{ij} - \hat{\chi}^i\hat{\chi}^j, \qquad k_\parallel \equiv \mathbf{k}\cdot\hat{\chi}, \quad k_\perp^2 \equiv k^2 - k_\parallel^2. \tag{1.10}$$

Rotational symmetry around $\hat{\chi}$ implies that the principal gradient block can be written as

$$\mathsf{Z}^{ij}\,k_i k_j \;=\; Z_\parallel\,k_\parallel^2 \;+\; Z_\perp\,k_\perp^2 \;+\; \cdots, \tag{1.11}$$

with possible small off–diagonal mixings suppressed by the residual symmetries. After canonical normalization one obtains anisotropic sound cones for conservative sectors,

$$\omega^2 \;=\; c_\parallel^2\,k_\parallel^2 \;+\; c_\perp^2\,k_\perp^2 \;+\; m_{\text{eff}}^2 \;+\; \cdots, \qquad 0 < c_\perp^2 \leq c_\parallel^2 \leq 1, \tag{1.12}$$

ensuring hyperbolicity and subluminality within the $g^{(0)}$ light cone.

Generalized dispersion and instability band

Seeking plane waves $\delta\mathbf{S} \propto e^{i\mathbf{k}\cdot\mathbf{x}+\lambda t}$, the normal modes solve

$$\det\Big[\ \lambda^2\mathsf{K} \ + \ \lambda\mathsf{C} \ + \ \mathsf{Z}^{ij}k_ik_j \ + \ \mathsf{M}^2\ \Big] \ = \ 0. \tag{1.13}$$

For small $|\mathbf{k}|$ the largest eigenvalue typically behaves as

$$\lambda_{\max}(\mathbf{k}) \ = \ \lambda_0 \ - \ D_{\parallel}\,k_{\parallel}^2 \ - \ D_{\perp}\,k_{\perp}^2 \ + \ \mathcal{O}(k^4), \tag{1.14}$$

where λ_0 encodes homogeneous growth/decay from $(\mathsf{K},\mathsf{C},\mathsf{M}^2)$ and $(D_{\parallel}, D_{\perp})$ are spectral diffusivities extracted from Z^{ij}. If $\lambda_0 > 0$, a *modulational instability* selects a band of wavenumbers

$$k_{\text{peak}}^2 \ \simeq \ \arg\max_{\mathbf{k}}\ \lambda_{\max}(\mathbf{k}) \ \sim \ \begin{cases} \lambda_0/(2D_{\parallel}) & (\mathbf{k}\parallel\hat{\chi}), \\ \lambda_0/(2D_{\perp}) & (\mathbf{k}\perp\hat{\chi}), \end{cases}$$

defining a seed scale $\xi \sim k_{\text{peak}}^{-1}$ for subsequent structure (defects and resonance; Secs. **??**, **??**).

Background Raychaudhuri relation (emergent frame)

Projecting the effective Einstein equation of the emergent geometry (Vol. I, Ch. 3) onto the u^μ foliation yields

$$\dot{H} \ = \ -\frac{1}{2M_{\text{eff}}^2(\rho_0)}\left(\rho_{\text{tot}} + p_{\text{tot}}\right) \ - \ \sigma^2 + \omega^2 \ + \ \cdots, \tag{1.15}$$

with shear σ^2, vorticity ω^2, and coarse–grained $(\rho_{\text{tot}}, p_{\text{tot}})$ including the η–sector. The quiescent void has $H = 0$; a quasi–de Sitter phase appears only when the η sector coherently dominates (Sec. **??**), and observational bounds enforce $|\sigma|, |\omega| \ll H$ during that epoch.

Gauge and constraints

In comoving–aether gauge ($\delta T = 0$), scalar, vector, and tensor modes are defined with respect to γ_{ij}. Residual internal gauge symmetries (fiber isometries) imply linear constraints (Ward identities) that eliminate spurious dof; they are enforced at the quadratic level as first–class constraints in the Hamiltonian form of (1.7). Tensor modes are subsequently put in TT gauge for power–spectrum computations.

Outcome. Eqs. (1.7)–(1.13) provide the *pre–metric* linear response of the aether around S_0. They encode: (i) the conditions for the onset of dynamics (sign of λ_0), (ii) the anisotropic seed scale ξ selected by $(D_{\parallel}, D_{\perp})$, (iii) the causal sound cones (2.7), and (iv) the thermodynamic arrow (2.8). This will be the staging ground for the spectral–viscosity–driven inflationary phase (Sec. **??**) and for the quantitative predictions of the primordial spectra (Sec. **??**).

Chapter 2

Linear Stability and the Emergence of Time/Arrow

2.1 Quadratic Expansion of the Unified Action

We expand the unified spectral action of Volume I (structural + matter + Palatini-like curvature) about the homogeneous, ordered reference state

$$S_0 = \{\rho_0,\ \eta_0 > 0,\ \Phi_0 = 0,\ \chi^{(0)\mu},\ T_0\}, \qquad g^{(0)}_{\mu\nu} = g_{\text{eff}}[S_0], \qquad u^\mu \text{ unit timelike.}$$

Let the perturbation multiplet be

$$\delta\mathbf{S} \equiv \big(\delta\rho,\ \delta\eta,\ \delta\alpha,\ \delta\chi_I,\ \delta\Phi,\ \delta e^a{}_\mu, \dots\big)^\top.$$

To quadratic order (after integrations by parts and fixing background gauges: comoving–aether $\delta T = 0$, and TT for tensors later) the Lagrangian can be written as

$$\mathcal{L}^{(2)} = \tfrac{1}{2}\,\delta\mathbf{S}^\top\Big[\ \mathsf{K}\,\partial_t^2\ +\ \mathsf{C}\,\partial_t\ +\ \mathsf{Z}^{ij}\nabla_i\nabla_j\ +\ \mathsf{M}^2\ \Big]\,\delta\mathbf{S}, \tag{2.1}$$

where all coefficient operators $\mathsf{K}, \mathsf{C}, \mathsf{Z}^{ij}, \mathsf{M}^2$ are background tensors built from the spectral derivative D_μ and the emergent connection ω_μ (Vol. I, Ch. 2–3), evaluated at S_0.

No-ghost gate. Positivity of the time-kinetic block

$$\mathsf{K} \succ 0 \tag{2.2}$$

is required for all propagating combinations to ensure positive energy and a well-defined Hamiltonian.

Block structure and anisotropy. Splitting momenta as $k_\parallel \equiv \mathbf{k}\cdot\hat{\chi}^{(0)}$ and $k_\perp^2 = k^2 - k_\parallel^2$, the principal-gradient block reads

$$\mathsf{Z}^{ij}k_ik_j = \begin{pmatrix} Z_{\text{sc},\parallel}\,k_\parallel^2 + Z_{\text{sc},\perp}\,k_\perp^2 & * \\ * & Z_{\text{tr},\parallel}\,k_\parallel^2 + Z_{\text{tr},\perp}\,k_\perp^2 \end{pmatrix}, \tag{2.3}$$

exhibiting the mild anisotropy inherited from the internal director $\hat{\chi}^{(0)}$.

2.2 Principal Symbol, Normal Modes, and Seed Scales

Seeking plane waves $\delta\mathbf{S} \propto e^{i\mathbf{k}\cdot\mathbf{x}+\lambda t}$, Eq. (2.1) gives the generalized eigenvalue problem

$$\det\Big[\ \lambda^2\mathsf{K} + \lambda\,\mathsf{C} + \mathsf{Z}^{ij}k_ik_j + \mathsf{M}^2\ \Big] = 0. \tag{2.4}$$

For small $|\mathbf{k}|$ the eigenvalues take the generic form

$$\lambda_\alpha(\mathbf{k}) = \lambda_{\alpha,0}\ -\ D_{\alpha,\parallel}\,k_\parallel^2\ -\ D_{\alpha,\perp}\,k_\perp^2\ +\ \mathcal{O}(k^4), \tag{2.5}$$

with $\lambda_{\alpha,0}$ the homogeneous growth/decay rates and $D_{\alpha,\parallel/\perp}$ the directional spectral diffusivities. A *modulational instability* is present if some $\lambda_{\max,0} > 0$, selecting a band around k_{peak} and defining a *seed scale*

$$\xi \sim k_{\text{peak}}^{-1}, \tag{2.6}$$

which controls both defect densities (Sec. **??**) and the resonance pattern (Sec. **??**).

Wave-like sectors. In conservative leading sectors one obtains anisotropic sound-cone relations

$$\omega^2 = c_\parallel^2\, k_\parallel^2 + c_\perp^2\, k_\perp^2 + m_{\text{eff}}^2 + \cdots, \qquad 0 < c_\perp^2 \le c_\parallel^2 \le 1, \tag{2.7}$$

with hyperbolicity and subluminality measured relative to $g_{\text{eff}}^{(0)}$.

2.3 Activation of Temporal Ordering T and Arrow η

The internal ordering field T supplies a *clock variable* once the instability activates dynamical channels. The spectral viscosity $\eta > 0$ opens dissipative pathways, selecting a preferred direction by nonnegative entropy production. Coarse-grained in the aether frame u^μ, the entropy balance reads

$$\partial_\mu(s\, u^\mu) = \sigma_{\text{diss}} \;=\; \sum_A \gamma_A\, \Pi_A^2 \;\ge 0, \tag{2.8}$$

with generalized forces Π_A (e.g. strain rates, phase–gradient fluxes) and positive transport coefficients γ_A controlled by η (cf. Vol. I). Thus:

> *Existence of time* (ordering) is attributed to T becoming a valid internal clock when dynamics turns on; *direction of time* (arrow) is set by η through irreversible production $\sigma_{\text{diss}} \ge 0$. With T but $\eta = 0$ evolution can be reversible; with T and $\eta > 0$ a preferred direction is dynamically selected.

2.4 Hyperbolicity, No-Ghosts, and Early-Time Gates

The early-universe background must satisfy the following *consistency gates*, adapted from Volume I:

- **(G1) No ghosts:** $\mathsf{K} \succ 0$ [Eq. (2.2)].
- **(G2) Hyperbolicity & causality:** the principal cones defined by Z^{ij} lie inside the light cone of $g_{\text{eff}}^{(0)}$; in particular $0 < c_T^2,\ c_s^2(\theta) \le 1$ for tensor/scalar sectors.
- **(G3) Entropy production:** $\sigma_{\text{diss}} \ge 0$ with $\eta_0 > 0$ [Eq. (2.8)].
- **(G4) Controlled anisotropy:** shear/vorticity of u^μ obey $|\sigma|/H,\ |\omega|/H \ll 1$ so statistical anisotropy is within observational bounds.
- **(G5) Restoration:** when the η coherence decays, the dynamics reduces to adiabatic scalar perturbations with $c_T \to 1$ and standard propagation.

Outcome. With these gates satisfied, the linear analysis fixes (i) the existence and scale of the instability that seeds structure [Eqs. (2.4)–(2.6)], (ii) the emergence of a clock and an arrow via (T, η) [Eq. (2.8)], and (iii) the admissible parameter domain for the subsequent inflating stage. This completes the foundational chapter and prepares the ground for Chap. **??**.

Chapter 3

Inflation as a Spectral Viscosity Wave (No Inflaton)

3.1 Background Equations for η and the Emergent Hubble Rate

Effective action, emergent Einstein equations, and energy balance

We consider the total action (Vol. I)

$$S_{\rm tot} \;=\; \int d^4x\,\sqrt{-g_{\rm eff}}\,\Big[\,\mathcal{L}_{\rm struct}(\mathcal{S}, D_\mu\mathcal{S}) \;+\; \mathcal{L}_{\rm curv}(g_{\rm eff},\omega_\mu) \;+\; \mathcal{L}_\eta(\eta; g_{\rm eff}) \;+\; \mathcal{L}_{\rm int}\Big], \tag{3.1}$$

with the η-sector

$$\mathcal{L}_\eta = \frac{\kappa_\eta(\mathcal{S})}{2}\, g^{\mu\nu}_{\rm eff}\nabla_\mu\eta\nabla_\nu\eta - V_\eta(\eta) - \frac{\Gamma_\eta(\mathcal{S})}{2}\,(u^\mu\nabla_\mu\eta)^2 \;+\; \dots \tag{3.2}$$

where dots denote higher-derivative operators suppressed by the consistency gates of Vol. I. Variation w.r.t. $g^{\mu\nu}_{\rm eff}$ defines $T_{\mu\nu}$ and the emergent Einstein equation

$$M^2_{\rm eff}(\rho)\,G_{\mu\nu}[g_{\rm eff}] \;=\; T^{(\eta)}_{\mu\nu} + T^{(\rm struct)}_{\mu\nu} + T^{(\rm matt)}_{\mu\nu}. \tag{3.3}$$

In the aether rest frame u^μ with homogeneous background,

$$\rho_\eta = \tfrac{1}{2}\kappa_\eta\dot\eta^2 + V_\eta, \qquad p_\eta = \tfrac{1}{2}\kappa_\eta\dot\eta^2 - V_\eta, \qquad w_\eta = \frac{p_\eta}{\rho_\eta}. \tag{3.4}$$

Projecting (3.3) along u^μ yields the effective Friedmann–Raychaudhuri system

$$3M^2_{\rm eff}H^2 = \rho_{\rm tot}, \qquad -2M^2_{\rm eff}\dot H = \rho_{\rm tot} + p_{\rm tot} \;-\; 2M_{\rm eff}\dot M_{\rm eff}\,H, \tag{3.5}$$

where the last term stems from the time dependence of $M_{\rm eff}(\rho)$ (Vol. I, Ch. 2). In regimes where $|\dot M_{\rm eff}|/(HM_{\rm eff}) \ll 1$, the standard-looking forms are recovered to leading order.

Dissipative microphysics implies the covariant balance equations

$$\dot\rho_\eta + 3H(\rho_\eta + p_\eta) = -\,\Gamma_\eta\,\dot\eta^2 \;+\; \mathcal{O}(\dot\kappa_\eta\,\dot\eta^2), \tag{3.6}$$

$$\dot\rho_X + 3H(\rho_X + p_X) = +\,\Gamma_\eta\,\dot\eta^2\,, \tag{3.7}$$

ensuring total energy conservation and nonnegative entropy production (Vol. I gate).

Field equation, dissipative slow roll, and validity domain

Varying (3.2) w.r.t. η gives

$$\kappa_\eta\ddot\eta + (3H\kappa_\eta + \Gamma_\eta)\dot\eta + \tfrac{1}{2}\dot\kappa_\eta\,\dot\eta + V'_\eta(\eta) = 0. \tag{3.8}$$

Define the spectral slow parameters

$$\epsilon_\eta \equiv -\frac{\dot{H}}{H^2}, \qquad \eta_\eta \equiv \frac{\dot{\epsilon}_\eta}{H\epsilon_\eta}, \qquad \varsigma_M \equiv \frac{\dot{M}_{\rm eff}}{HM_{\rm eff}}, \qquad \varsigma_\kappa \equiv \frac{\dot{\kappa}_\eta}{H\kappa_\eta}. \tag{3.9}$$

The *dissipative slow–roll* regime is

$$\epsilon_\eta \ll 1, \qquad |\eta_\eta| \ll 1, \qquad |\varsigma_M|, |\varsigma_\kappa| \ll 1, \qquad \ddot{\eta} \ll (3H + \Gamma_\eta/\kappa_\eta)\dot{\eta}. \tag{3.10}$$

Then

$$(3H\kappa_\eta + \Gamma_\eta)\dot{\eta} \simeq -V'_\eta(\eta), \qquad H^2 \simeq \frac{V_\eta}{3M_{\rm eff}^2}, \tag{3.11}$$

and the Hubble slow parameter becomes

$$\epsilon_\eta \simeq \frac{\kappa_\eta M_{\rm eff}^2}{2}\left(\frac{V'_\eta}{V_\eta}\right)^2\left(\frac{3H}{3H + \Gamma_\eta/\kappa_\eta}\right)^2 + \mathcal{O}(\varsigma_M, \varsigma_\kappa). \tag{3.12}$$

The number of e–folds between η_i and η_f is

$$N = \int_{t_i}^{t_f} H\, dt \simeq \frac{1}{M_{\rm eff}^2}\int_{\eta_f}^{\eta_i} \frac{V_\eta}{\kappa_\eta V'_\eta}\left(\frac{3H + \Gamma_\eta/\kappa_\eta}{3H}\right) d\eta. \tag{3.13}$$

Hamilton–Jacobi (single-clock) formulation

When η is monotonic, one can view H as a function of η. Combining (3.5) and (3.8) gives

$$\dot{\eta} \;=\; -\frac{2M_{\rm eff}^2}{\kappa_\eta}\frac{H_{,\eta}}{1+\Upsilon}, \qquad \Upsilon \equiv \frac{\Gamma_\eta}{3H\kappa_\eta}, \tag{3.14}$$

and the Hamilton–Jacobi constraint

$$V_\eta(\eta) \;=\; 3M_{\rm eff}^2 H^2 \;-\; \frac{2M_{\rm eff}^4}{\kappa_\eta}\frac{H_{,\eta}^2}{(1+\Upsilon)^2} \;+\; \mathcal{O}(\varsigma_M, \varsigma_\kappa). \tag{3.15}$$

The slow parameter is

$$\epsilon_\eta \;=\; \frac{2M_{\rm eff}^2}{\kappa_\eta}\frac{H_{,\eta}^2}{H^2(1+\Upsilon)^2} \;+\; \mathcal{O}(\varsigma_M, \varsigma_\kappa), \tag{3.16}$$

and the e–folds follow from

$$N(\eta) = \int_{\eta_{\rm end}}^{\eta} \frac{H}{\dot{\eta}}\, d\eta \;=\; -\int_{\eta_{\rm end}}^{\eta} \frac{\kappa_\eta}{2M_{\rm eff}^2}\frac{(1+\Upsilon)H}{H_{,\eta}}\, d\eta. \tag{3.17}$$

Eqs. (3.15)–(3.17) are convenient to scan model classes without solving time evolution explicitly.

Consistency gates and background admissibility

The early–time gates adapted from Vol. I require:

1. **No ghosts:** $\kappa_\eta > 0$ and positive time–kinetic block for all propagating combinations.
2. **Causality/hyperbolicity:** principal scalar/tensor cones inside the $g_{\rm eff}$ cone; $0 < c_s^2(\theta), c_T^2 \leq 1$.
3. **Controlled running:** $|\varsigma_M|, |\varsigma_\kappa| \ll 1$ during inflation so that backreaction from $M_{\rm eff}(\rho)$ and κ_η does not spoil quasi–de Sitter.
4. **Entropy production:** $\Gamma_\eta \geq 0$ ensures $\sigma_{\rm diss} \geq 0$.

Exit criteria, reheating scale, and matching

Inflation ends when $\epsilon_\eta(\eta_{\rm end}) = 1$ or when the slow–roll conditions break. Near exit, the friction–dominated solution crosses to an oscillatory/overdamped phase depending on $V''_\eta(\eta_{\rm end})$ and $\Upsilon_{\rm end}$. The effective reheating temperature is

$$T_{\rm reh} \simeq \left(\frac{90}{\pi^2 g_\star}\right)^{1/4} (M_{\rm eff}\Gamma_{\rm eff})^{1/2}, \qquad \Gamma_{\rm eff} \equiv \Gamma_\eta/\kappa_\eta, \tag{3.18}$$

and the restoration gate demands a smooth match to an adiabatic hot phase with $c_T \to 1$ and decaying anisotropies.

Summary. Eqs. (3.8)–(3.17) provide a complete, model–agnostic backbone for η–driven quasi–de Sitter dynamics in the emergent geometry. Dissipation (Γ_η), running normalizations ($M_{\rm eff}, \kappa_\eta$), and internal anisotropy enter transparently, while the consistency gates delimit the admissible background domain.

3.2 Model Classes for $V_\eta(\eta)$ and the Duration of Inflation

We organize viable η-wave models by the shape of $V_\eta(\eta)$ in the dissipative slow–roll regime (3.10), using the background relations (3.11)–(3.13). Throughout this section we take $M_{\rm eff}$ and κ_η slowly varying (their fractional runnings $\varsigma_M, \varsigma_\kappa \ll 1$), and define the *dissipation ratio*

$$\Upsilon \equiv \frac{\Gamma_\eta}{3H\,\kappa_\eta}, \qquad (1+\Upsilon) > 0.$$

General slow–roll algebra

Given a monotonic branch of η, the Hubble slow parameter and e–folds follow from

$$\epsilon_\eta \simeq \frac{\kappa_\eta M_{\rm eff}^2}{2}\left(\frac{V'_\eta}{V_\eta}\right)^2 \frac{1}{(1+\Upsilon)^2}, \qquad N(\eta_i \to \eta_f) \simeq \frac{1+\Upsilon}{M_{\rm eff}^2\kappa_\eta}\int_{\eta_f}^{\eta_i} \frac{V_\eta}{V'_\eta}\, d\eta. \tag{3.19}$$

When the mapping $\eta \leftrightarrow N$ is invertible, one obtains

$$n_s - 1 \simeq -2\epsilon_\eta - \eta_\eta - s_c, \qquad r \simeq 16\,\epsilon_\eta\,\frac{c_s}{c_T},$$

with the angle-averaged c_s (Sec. **??**) and $s_c \equiv \dot c_s/(Hc_s)$ small.

3.2.1 Plateau class (Starobinsky/T-model–like)

Asymptotically flat potentials yield nearly universal predictions. Two representatives are

$$V_\eta(\eta) = V_0\Big(1 - e^{-\eta/\eta_\star}\Big)^2, \qquad V_\eta(\eta) = V_0\,\tanh^2\frac{\eta}{\eta_\star}. \tag{3.20}$$

In the large–η regime, $V'_\eta/V_\eta \simeq 2e^{-\eta/\eta_\star}/\eta_\star$, and (3.19) gives

$$N \simeq \frac{1+\Upsilon}{M_{\rm eff}^2\kappa_\eta}\,\frac{\eta_\star^2}{2}\,e^{\eta/\eta_\star}, \qquad \Longrightarrow \qquad e^{-\eta/\eta_\star} \simeq \frac{\eta_\star^2}{2M_{\rm eff}^2\kappa_\eta}\,\frac{1+\Upsilon}{N}. \tag{3.21}$$

Hence the slow parameter becomes

$$\epsilon_\eta \simeq \frac{\eta_\star^2}{2\,\kappa_\eta M_{\rm eff}^2}\,\frac{1}{N^2} + \mathcal{O}(N^{-3}), \tag{3.22}$$

independent of Υ at leading order (the Υ-dependence cancels between (3.19) and (3.21)). Consequently,

$$n_s - 1 \simeq -\frac{2}{N} + \mathcal{O}(N^{-2}, \varsigma_{M,\kappa}, s_c), \qquad r \simeq \frac{8\,\eta_\star^2}{\kappa_\eta M_{\rm eff}^2}\,\frac{1}{N^2}\,\frac{c_s}{c_T}, \tag{3.23}$$

a robust $1/N^2$ suppression for r with a model–dependent prefactor set by the spectral normalizations.

3.2.2 Hilltop class (near-maximum slow roll)

Let

$$V_\eta(\eta) = V_0\Big[1 - \big(\tfrac{\eta}{\mu}\big)^p + \cdots\Big], \qquad p > 2,\ |\eta| \ll \mu. \tag{3.24}$$

Then $V'_\eta/V_\eta \simeq -\,p\,\eta^{p-1}/\mu^p$, and (3.19) yields

$$N \;\simeq\; \frac{1+\Upsilon}{M_{\rm eff}^2\kappa_\eta}\,\frac{\mu^2}{p(p-2)}\,\Big(\frac{\mu}{\eta}\Big)^{p-2}, \qquad (p>2). \tag{3.25}$$

Inverting, one finds the scaling

$$\epsilon_\eta \;\simeq\; \frac{\kappa_\eta M_{\rm eff}^2\,p^2}{2\mu^2}\,\Big(\frac{\eta}{\mu}\Big)^{2(p-1)}\,\frac{1}{(1+\Upsilon)^2} \;\propto\; \frac{1}{(1+\Upsilon)^{\frac{2(p-1)}{p-2}}}\,N^{-\frac{2(p-1)}{p-2}}, \tag{3.26}$$

up to $\mathcal{O}(\varsigma_{M,\kappa})$. Therefore

$$n_s - 1 \;\simeq\; -\,\frac{2(p-1)}{(p-2)}\,\frac{1}{N} \;+\; \mathcal{O}\Big(N^{-2(p-1)/(p-2)}, \varsigma_{M,\kappa}, s_c\Big), \qquad r \;\sim\; \mathcal{O}\Big(N^{-2(p-1)/(p-2)}\Big)\,\frac{c_s}{c_T}\,, \tag{3.27}$$

with additional suppression from dissipation through the prefactor in (3.26). The case $p = 4$ gives the familiar $(n_s - 1) \simeq -3/N$ and $r \propto N^{-3}$ behaviour.

3.2.3 Monomial/attractor class

For $V_\eta(\eta) = \lambda\,\eta^q$ with $q > 0$, we obtain

$$N \;\simeq\; \frac{1+\Upsilon}{M_{\rm eff}^2\kappa_\eta}\,\frac{\eta^2}{2q}\,, \qquad \epsilon_\eta \;\simeq\; \frac{q}{4N}\,\frac{1}{(1+\Upsilon)}\,, \tag{3.28}$$

and hence

$$n_s - 1 \;\simeq\; -\,\frac{q+2}{2N} \;+\; \mathcal{O}(N^{-2}, \varsigma_{M,\kappa}, s_c), \qquad r \;\simeq\; \frac{4q}{N}\,\frac{1}{(1+\Upsilon)}\,\frac{c_s}{c_T}. \tag{3.29}$$

Dissipation suppresses r by the factor $(1+\Upsilon)^{-1}$ and slightly delays the end of inflation relative to the conservative case.

3.2.4 Constant-roll / exponential potentials

For $V_\eta = V_0\,e^{-\lambda\eta}$, one finds a constant $\epsilon_\eta = (\kappa_\eta M_{\rm eff}^2\lambda^2/2)/(1+\Upsilon)$. Power-law inflation arises if $\epsilon_\eta < 1$. The spectral tilt is approximately $n_s - 1 \simeq -2\epsilon_\eta$ and $r \simeq 16\epsilon_\eta\,c_s/c_T$. Such models are tightly constrained unless Υ is sizeable.

3.2.5 Remarks on running normalizations and anisotropy

Slow runnings $\varsigma_M, \varsigma_\kappa$ shift n_s by $\mathcal{O}(\varsigma_{M,\kappa})$, and affect r through the prefactors in (3.23), (3.26), (3.29). Mild statistical anisotropy enters via the angle dependence of the scalar sound speed $c_s(\theta)$ and is constrained by $|c_\parallel - c_\perp| \ll 1$ (cf. Sec. **??**). These corrections can be kept below current bounds while retaining the leading N–scalings above.

Summary. Plateau models predict $n_s - 1 \simeq -2/N$ and $r \propto N^{-2}$ with a normalization set by $(\kappa_\eta, M_{\rm eff}, c_s/c_T)$; hilltops give steeper $1/N$ tilts and strongly suppressed r; monomials yield the familiar $(n_s - 1) \simeq -(q+2)/(2N)$ with $r \propto 1/[N(1+\Upsilon)]$. These relations provide clean targets to confront the η–wave paradigm with data once $(N, \Upsilon, \kappa_\eta, M_{\rm eff})$ are fixed by the background.

3.3 Graceful Exit and Reheating into Φ and Structural Modes

Inflation driven by the η–wave must end smoothly and hand its energy to matterlike excitations of the spectral phase Φ and to structural sectors $(\rho, \chi^\mu, \ldots)$, while restoring the late–time GR–like propagation and adiabatic scalar perturbations. We develop a model–independent treatment based on the effective Lagrangian of Sec. **??** and on coarse–grained conservation equations in the aether frame u^μ.

3.3.1 Exit trigger: end of slow roll and onset of oscillations

Inflation terminates when $\epsilon_\eta \to \mathcal{O}(1)$ [cf. Eq. (**??**)], either because V_η steepens or because the kinetic fraction $\kappa_\eta \dot{\bar{\eta}}^2/(2V_\eta)$ grows. Near a minimum $\eta_{\rm min}$ with $V_\eta(\eta) \simeq V_{\rm vac} + \frac{1}{2}m_\eta^2(\eta-\eta_{\rm min})^2 + \cdots$, the homogeneous condensate obeys

$$\ddot{\bar{\eta}} + 3H\dot{\bar{\eta}} + m_{\eta,\rm eff}^2\,(\bar{\eta}-\eta_{\rm min}) + \frac{\Gamma_\eta}{\kappa_\eta}\,\dot{\bar{\eta}} \;\simeq\; 0, \qquad m_{\eta,\rm eff}^2 \equiv V_\eta''(\eta_{\rm min}) + \mathcal{O}(H\dot{\bar{\eta}}), \tag{3.30}$$

leading to damped oscillations with frequency $m_{\eta,\rm eff}$ and amplitude $A(t) \propto a^{-3/2}\exp\!\big[-\frac{1}{2}\int(\Gamma_\eta/\kappa_\eta)dt\big]$. These oscillations are the engine of particle production.

3.3.2 Energy–transfer channels and coarse–grained reheating

We group the post–inflationary components into ρ_η (coherent condensate), ρ_R (radiationlike bath sourced by Φ and relativistic structural modes), and ρ_S (nonrelativistic or quasi–coherent structural excitations). Their continuity equations read

$$\dot{\rho}_\eta + 3H(\rho_\eta + p_\eta) = -\,\Gamma_{\rm tot}\,\dot{\bar{\eta}}^2, \tag{3.31}$$

$$\dot{\rho}_R + 4H\rho_R = +\,\Gamma_R\,\dot{\bar{\eta}}^2 + \mathcal{C}_{\rm th}, \tag{3.32}$$

$$\dot{\rho}_S + 3H\rho_S = +\,\Gamma_S\,\dot{\bar{\eta}}^2 - \mathcal{C}_{\rm th}, \tag{3.33}$$

with $\Gamma_{\rm tot} = \Gamma_R + \Gamma_S$, and $\mathcal{C}_{\rm th}$ describing conversion between nonrelativistic structural quanta and the thermal bath. Equations (3.31)–(3.33) follow from the Rayleigh term and the interaction vertices

$$\mathcal{L}_{\rm int} \;\supset\; \frac{g_{\eta\Phi}}{2}\,\eta\,|\Phi|^2 \;+\; \frac{h_{\eta S}}{2}\,\eta\,\mathcal{I}[S] \;+\; \frac{\lambda_{\eta\Phi}}{4}\,\eta^2|\Phi|^2 \;+\; \frac{\lambda_{\eta S}}{4}\,\eta^2\mathcal{I}[S] \;+\; \cdots, \tag{3.34}$$

where $\mathcal{I}[S]$ denotes the leading quadratic invariant of the chosen structural channel (e.g. $(\delta\rho)^2$, $(\delta\chi)^2$).

3.3.3 Perturbative decay widths

Whenever kinematics allows and the couplings are weak, the condensate can be treated as quanta of mass $m_{\eta,\rm eff}$ decaying with widths

$$\Gamma(\eta\to\Phi\Phi) = \frac{g_{\eta\Phi}^2}{32\pi\,\kappa_\eta\,m_{\eta,\rm eff}}\sqrt{1-\frac{4m_\Phi^2}{m_{\eta,\rm eff}^2}}\,, \tag{3.35}$$

$$\Gamma(\eta\to SS) = \frac{h_{\eta S}^2}{32\pi\,\kappa_\eta\,m_{\eta,\rm eff}}\sqrt{1-\frac{4m_S^2}{m_{\eta,\rm eff}^2}}\,, \tag{3.36}$$

with κ_η from the kinetic normalization [cf. Eq. (**??**)]. The total perturbative rate contributes to Γ_R, Γ_S in Eqs. (3.32)–(3.33).

3.3.4 Nonperturbative preheating and Floquet analysis

For trilinear or quartic couplings in (3.34), the mode functions X_k of a daughter field ($X = \Phi$ or a structural fluctuation) satisfy

$$\ddot{X}_k + 3H\dot{X}_k + \Big[\omega_k^2 + q_X\,m_{\eta,\rm eff}^2\cos(2m_{\eta,\rm eff}t) + \cdots\Big]X_k = 0, \qquad \omega_k^2 \equiv \frac{k^2}{a^2} + m_X^2 + \delta m_X^2, \tag{3.37}$$

where q_X encodes the periodic driving (e.g. $q_\Phi = \lambda_{\eta\Phi}A^2/(4m_{\eta,\rm eff}^2)$ for a quartic coupling, with A the oscillation amplitude). Neglecting expansion on sub–Hubble time scales maps (3.37) to a Mathieu/Hill equation with Floquet exponent μ_k; broad resonance occurs for $q_X \gtrsim \mathcal{O}(1)$, leading to $n_k \propto e^{2\mu_k m_{\eta,\rm eff}t}$ until backreaction shuts it off at $\lambda_{\eta X}\langle X^2\rangle \sim m_{\eta,\rm eff}^2$ or when H redshifts modes out of resonance bands. Structural channels often preheat efficiently (large q_S) because their effective masses arise from the same sector that controls $m_{\eta,\rm eff}$.

3.3.5 Restoration gate and matching to the hot epoch

A consistent exit obeys the following *restoration* conditions:

1. **(R1) Adiabaticity:** the multi–fluid system relaxes to a single adiabatic mode on super–Hubble scales; isocurvature sourced by structural preheating decays, $\mathcal{S}_{XY} \to 0$, or remains below observational bounds.

2. **(R2) Propagation:** the effective tensor speed returns to unity and statistical anisotropy is suppressed ($c_T \to 1$, $|c_\parallel - c_\perp| \ll 1$).

3. **(R3) Thermalization:** microscopic scattering in the Φ/S bath yields a common temperature T on a timescale $\tau_{\rm th} \ll H^{-1}$.

When these are satisfied, the system is well approximated by a radiation fluid with $\rho_R = \frac{\pi^2}{30}\, g_\star(T)\, T^4$ and negligible residual anisotropy.

3.3.6 Reheating temperature and duration

If reheating is dominated by perturbative decay with effective rate $\Gamma_{\rm tot}$, the end of reheating occurs when $H \simeq \Gamma_{\rm tot}$. Using the emergent Friedmann relation (**??**) with $M_{\rm eff}$ at the end of reheating,

$$T_{\rm rh} \;\simeq\; \left(\frac{90}{\pi^2 g_\star(T_{\rm rh})}\right)^{1/4} \left(M_{\rm eff}\,\Gamma_{\rm tot}\right)^{1/2}, \tag{3.38}$$

generalizing the standard expression by replacing $M_P \to M_{\rm eff}$. For preheating–dominated transfer, $T_{\rm rh}$ is set by the energy density at the end of the resonant stage minus backreaction losses; Eq. (3.38) still provides a useful proxy with an effective $\Gamma_{\rm tot} \sim \mu\, m_{\eta,\rm eff}$.

The number of e–folds between horizon exit of a CMB pivot mode and the end of inflation depends on the reheating history:

$$N_\star \simeq 57.5 - \ln\!\left(\frac{k_\star}{0.05\,{\rm Mpc}^{-1}}\right) + \frac{1}{4}\ln\!\left(\frac{V_\star}{M_{\rm eff}^4}\right) - \frac{1-3w_{\rm reh}}{12(1+w_{\rm reh})}\,\ln\!\left(\frac{\rho_{\rm reh}}{\rho_{\rm end}}\right), \tag{3.39}$$

where $w_{\rm reh}$ is the mean reheating EoS, and $V_\star$ is the inflationary potential at $k_\star$ (all in the emergent frame).

3.3.7 Spectral imprints from exit and reheating

The exit stage can leave subleading but testable imprints:

- **Non–Gaussianity:** bursts of particle production (especially in structural channels) can generate localized features in the bispectrum. The amplitude scales as $f_{\rm NL}^{\rm exit} \sim \mathcal{O}(q_X\,\Delta N_{\rm exit})$ suppressed if $\Delta N_{\rm exit} \ll 1$ and adiabaticity is quickly restored.

- **Isocurvature:** inefficient thermalization of structural relics yields $S_{XR} \neq 0$; the restoration gate (R1) bounds the couplings to ensure $|S_{XR}| \ll 10^{-5}$ at CMB scales.

- **Gravitational waves:** anisotropic stresses from violent preheating can source a high–frequency stochastic background with peak frequency $f_{\rm pk} \sim 0.1\, m_{\eta,\rm eff}/(2\pi a_0)$, subject to BBN/CMB constraints.

Summary. A graceful exit in the η–wave paradigm proceeds through damped oscillations of the viscosity condensate, with energy redistributed to Φ and structural modes by a combination of perturbative decays [Eqs. (3.35)–(3.36)] and nonperturbative preheating [Eq. (3.37)]. Provided the restoration conditions (R1)–(R3) are met, the system matches onto a hot, nearly isotropic epoch with reheating temperature (3.38). Residual imprints are controlled and furnish observational handles on the couplings $(g_{\eta\Phi}, h_{\eta S}, \lambda_{\eta X})$ and on the spectral normalization $M_{\rm eff}$.

3.4 Technical Machinery: Hamilton–Jacobi Flow, Mode Functions, and Open-System Reheating

This section gathers the core derivations underpinning the η–wave paradigm: a Hamilton–Jacobi (HJ) formulation with state-dependent normalization and friction, the canonical quantization of scalar/tensor modes with mild anisotropy, a Floquet analysis of preheating, and a GKSL open-system framework for decoherence and entropy production. These tools justify the background and exit dynamics of Secs. **??**–3.3 and prepare the ground for the observable predictions of Sec. 3.5.

3.4.1 Hamilton–Jacobi slow flow with $M_{\rm eff}(\rho)$ and Γ_η

Let $H \equiv \frac{1}{3}\nabla_\mu u^\mu$ be the emergent Hubble rate and assume a single-clock background $\bar\eta(t)$ controlling inflation. In the slow regime, Eqs. (**??**) and (**??**) imply a first-order flow for $H(\bar\eta)$. Write[1]

$$3\,M_{\rm eff}^2(\rho)\,H^2(\bar\eta) \;\simeq\; V_\eta(\bar\eta), \qquad (3H\,\kappa_\eta + \Gamma_\eta)\,\dot{\bar\eta} \;\simeq\; -\,V_\eta'(\bar\eta). \tag{3.40}$$

Differentiating the first relation with respect to $\bar\eta$ and combining with the second yields a generalized HJ equation:

$$\dot{\bar\eta} \;\simeq\; -\,\frac{2\,M_{\rm eff}^2}{\kappa_\eta}\,\frac{H'(\bar\eta)}{1+\Gamma_\eta/(3H\kappa_\eta)}\,, \qquad \epsilon_\eta \equiv -\frac{\dot H}{H^2} \;\simeq\; 2\,M_{\rm eff}^2\left(\frac{H'}{H}\right)^2\left(\frac{1}{1+\Gamma_\eta/(3H\kappa_\eta)}\right)^2. \tag{3.41}$$

The number of e-folds between $\bar\eta_i$ and $\bar\eta_f$ follows as

$$N(\bar\eta_i\to\bar\eta_f) \;\simeq\; \int_{\bar\eta_f}^{\bar\eta_i}\frac{H}{\dot{\bar\eta}}\,d\bar\eta \;=\; \int_{\bar\eta_f}^{\bar\eta_i}\frac{\kappa_\eta}{2M_{\rm eff}^2}\left(1+\frac{\Gamma_\eta}{3H\kappa_\eta}\right)\frac{H}{H'}\,d\bar\eta, \tag{3.42}$$

which reproduces (**??**) upon substituting $H^2 \simeq V_\eta/(3M_{\rm eff}^2)$.

3.4.2 Canonical normalization and the (mildly anisotropic) Mukhanov–Sasaki equation

In comoving-aether gauge ($\delta T = 0$ with vanishing momentum flux) define the curvature perturbation ζ on hypersurfaces orthogonal to u^μ. From the quadratic action (**??**) one introduces the canonically normalized variable $v \equiv z(\theta)\,\zeta$ with

$$z^2(\theta) \;=\; 2\,a^2\,\frac{M_{\rm eff}^2(\rho)\,\epsilon_\eta}{c_s^2(\theta)}\,, \qquad c_s^2(\theta) \;=\; c_\parallel^2\cos^2\theta + c_\perp^2\sin^2\theta, \tag{3.43}$$

where θ is the angle between $\mathbf{k}$ and $\hat\chi^{(0)}$ and $c_{\parallel,\perp}$ are the principal sound speeds extracted from (**??**). In conformal time $d\tau = dt/a$ the mode equation becomes

$$v_k'' + \left[c_s^2(\theta)\,k^2 - \frac{z''}{z}\right]v_k = 0, \qquad \frac{z''}{z} \;=\; a^2H^2\left(2-\epsilon_\eta+\tfrac{3}{2}\eta_\eta + s_c + \cdots\right), \tag{3.44}$$

with primes denoting $d/d\tau$ and $s_c \equiv \dot c_s/(Hc_s)$. Imposing Bunch–Davies initial conditions for $c_s k \gg aH$ yields the power (**??**). Angle-averaging over θ gives the $\langle s_c\rangle_\theta$ contribution quoted in (**??**).

Tensor sector. For h_{ij} one defines $u_{ij} = a\,M_{\rm eff}\,h_{ij}/2$ and obtains $u_{ij}'' + (c_T^2k^2 - a''/a)\,u_{ij} = 0$, with $c_T^2 = 1 + \delta c_T(\mathcal{S})$, leading to (**??**) and (**??**). Current bounds require $|\delta c_T| \ll 10^{-15}$ at late times; during inflation we assume the gate $|\delta c_T| \ll 1$.

[1] Corrections from $\dot M_{\rm eff} \neq 0$ are additive and suppressed by (**??**).

3.4.3 Floquet bands and backreaction in preheating

For a quartic vertex $\frac{\lambda_{\eta X}}{4}\eta^2 X^2$ and an oscillatory $\bar{\eta}(t) = A(t)\cos(m_{\eta,\mathrm{eff}}t)$, the comoving modes satisfy

$$X_k'' + \left[\omega_k^2(\tau) + 2q_X(\tau)\cos\!\big(2m_{\eta,\mathrm{eff}}\tau\big)\right] X_k = 0, \quad q_X(\tau) = \frac{\lambda_{\eta X}\, A^2(\tau)\, a^2(\tau)}{4\, m_{\eta,\mathrm{eff}}^2}, \tag{3.45}$$

with $\omega_k^2 = a^2(m_X^2 + \delta m_X^2) + k^2$. On sub-Hubble time scales (a nearly constant) this is a Mathieu/Hill equation. Instability bands with Floquet index $\mu_k > 0$ yield $n_k \propto e^{2\mu_k m_{\eta,\mathrm{eff}} t}$. Backreaction shuts resonance when $\lambda_{\eta X}\langle X^2\rangle \sim m_{\eta,\mathrm{eff}}^2$ or when Hubble drift moves modes out of the band. Trilinear couplings lead to narrow/broad bands depending on $g_{\eta X} A/m_{\eta,\mathrm{eff}}^2$.

3.4.4 GKSL open-system coarse graining and entropy production

Let ρ_{sys} be the reduced density matrix for long-wavelength modes $\{\zeta_{\mathbf{k}}, h_{ij,\mathbf{k}}\}$ after tracing out a short-scale environment (daughter fields and high-k modes) interacting through $\mathcal{L}_{\mathrm{int}}$ and the Rayleigh term in (**??**). Under the Born–Markov approximation one obtains a GKSL master equation

$$\dot{\rho}_{\mathrm{sys}} = -i[H_{\mathrm{eff}}, \rho_{\mathrm{sys}}] + \sum_a \left(L_a\, \rho_{\mathrm{sys}}\, L_a^\dagger - \tfrac{1}{2}\{L_a^\dagger L_a, \rho_{\mathrm{sys}}\}\right), \tag{3.46}$$

with Lindblad operators L_a dominated by combinations proportional to the coarse-grained $\dot{\eta}$ and to interaction vertices in (3.34). Positivity of the Kossakowski matrix ensures $\dot{S}_{\mathrm{vN}} \geq 0$, consistent with the macroscopic entropy production (2.8). The decoherence rate $\gamma_k \sim \Gamma_\eta + \mathcal{O}(g_{\eta X}^2, \lambda_{\eta X}^2)$ controls the decay of off-diagonal elements in the pointer basis (approximately field-amplitude eigenstates on super-Hubble scales), supporting the adiabatic restoration gate (R1).

3.4.5 Kinetic equilibration and thermalization

Given a bath dominated by Φ quanta with quartic coupling $\lambda_\Phi|\Phi|^4$ or contact interactions from (3.34), the $2 \to 2$ scattering cross section scales as $\sigma_{2\to 2} \sim \lambda_{\mathrm{eff}}^2/(16\pi\,\bar{s})$, with λ_{eff} an effective combination of couplings and $\bar{s} \sim T^2$ at temperature T. The thermalization rate is $\Gamma_{\mathrm{th}} \sim n\,\sigma_{2\to 2}\langle v\rangle \sim \lambda_{\mathrm{eff}}^2\, T$, implying $\tau_{\mathrm{th}} \sim (\lambda_{\mathrm{eff}}^2 T)^{-1}$. Requiring $\tau_{\mathrm{th}} \ll H^{-1}$ at $T \simeq T_{\mathrm{rh}}$ gives a lower bound on λ_{eff} consistent with (R3).

3.4.6 Matching conditions at the end of inflation

Across a hypersurface Σ defined by $\epsilon_\eta = 1$ one enforces: (i) continuity of the induced 3-metric of g_{eff} and of the aether 4-velocity u^μ projected on Σ; (ii) continuity of ζ and of its canonical momentum $\pi_\zeta = z^2\dot{\zeta}$ (up to negligible $\mathcal{O}(\Delta N_{\mathrm{exit}})$ sources). These conditions guarantee a smooth handover to the radiation-dominated epoch used in (3.39).

3.4.7 Consistency gates (technical layer)

- **(T1) No-ghost/positivity:** $\kappa_\eta > 0$, $\mathsf{K} \succ 0$ in all sectors.
- **(T2) Hyperbolicity/causality:** $0 < c_{\perp,\parallel}^2 \leq 1$, $c_T^2 \leq 1$.
- **(T3) Controlled anisotropy:** $|c_\parallel^2 - c_\perp^2| \ll 1$ during inflation.
- **(T4) Open-system consistency:** GKSL generator completely positive; $\dot{S}_{\mathrm{vN}} \geq 0$ matches macroscopic $\sigma_{\mathrm{diss}} \geq 0$.
- **(T5) Preheating safety:** backreaction halts resonance before violating (T1)–(T2); isocurvature damped to within observational bounds (R1).
- **(T6) Matching:** continuity of ζ, π_ζ across Σ and recovery of a single adiabatic mode.

Summary. Equations (3.41)–(3.42) provide a compact HJ control of the slow flow with dissipation; (11.7) fixes the scalar power with angle-dependent sound speed; (3.45) quantifies nonperturbative amplification and its shutdown; (3.46) frames decoherence and entropy growth consistently with the Rayleigh sector. These technical components close the logical loop between the background η dynamics, exit/reheating, and the observable spectra to be derived in Sec. 3.5.

3.5 Primary Predictions: (n_s, r) and Controlled Statistical Anisotropy

This section turns the η–wave framework into fitting-ready predictions. We assemble angle–averaged scalar and tensor spectra, controlled anisotropy signatures, non–Gaussianity estimates, and a minimal inversion map from observables to spectral parameters $(V_\eta, \kappa_\eta, M_{\rm eff}, \Gamma_\eta)$, together with operational consistency gates.

3.5.1 Scalar power, tilt, and running

From Secs. **??** and **??** the curvature spectrum at freeze-out is

$$\mathcal{P}_\zeta(k,\theta) = \frac{H_*^2}{8\pi^2\, M_{\rm eff}^2(\rho_*)\, \epsilon_{\eta,*}\, c_s(\theta)_*}\,, \qquad c_s^2(\theta) = c_\parallel^2 \cos^2\theta + c_\perp^2 \sin^2\theta, \tag{3.47}$$

with $*$ evaluated at horizon exit $k = a_* H_* / c_s(\theta)_*$. Angle–averaging ($\langle\cdot\rangle_\theta$) gives

$$\overline{\mathcal{P}}_\zeta(k) \equiv \langle \mathcal{P}_\zeta \rangle_\theta = \frac{H_*^2}{8\pi^2\, M_{{\rm eff},*}^2\, \epsilon_{\eta,*}\, \overline{c}_s}\,, \qquad \overline{c}_s^{\,-1} \equiv \langle c_s^{-1}(\theta)\rangle_\theta = \frac{1}{2}\int_0^\pi d\theta\, \sin\theta\, c_s^{-1}(\theta). \tag{3.48}$$

The spectral tilt and its running follow from slow variation of background quantities:

$$n_s - 1 \equiv \frac{d\ln \overline{\mathcal{P}}_\zeta}{d\ln k} \simeq -2\epsilon_\eta - \eta_\eta - s_c - \sigma_M, \tag{3.49}$$

$$\alpha_s \equiv \frac{dn_s}{d\ln k} \simeq -2\epsilon_\eta\eta_\eta - \xi_\eta^2 - \dot{s}_c/H - \dot{\sigma}_M/H, \tag{3.50}$$

with

$$\eta_\eta \equiv \frac{\dot{\epsilon}_\eta}{H\epsilon_\eta}, \quad s_c \equiv \frac{\dot{\overline{c}}_s}{H\,\overline{c}_s}, \quad \sigma_M \equiv \frac{d\ln M_{\rm eff}^2}{d\ln k} \simeq \frac{\dot{M}_{\rm eff}^2}{H\, M_{\rm eff}^2}, \quad \xi_\eta^2 \equiv \frac{M_{\rm eff}^4\, \kappa_\eta^2\, V_\eta' V_\eta'''}{V_\eta^2}\,,$$

where dots denote d/dt along the background. In the limit of constant $M_{\rm eff}$ and isotropic c_s one recovers the standard single–clock expressions.

Isocurvature. The single–clock gate requires isocurvature modes to be Boltzmann–suppressed or efficiently damped after reheating (Sec. 3.3). Operationally, one constrains any residual isocurvature fraction $\beta_{\rm iso} \equiv \mathcal{P}_{\rm iso}/(\mathcal{P}_{\rm ad} + \mathcal{P}_{\rm iso})$ to be sub-percent at CMB pivot scales; in the model this maps to bounds on couplings that source entropy modes during preheating (cf. Sec. 3.4.3).

3.5.2 Tensor sector and the tensor-to-scalar ratio

Tensor modes obey $c_T^2 = 1 + \delta c_T(\mathcal{S})$ with $|\delta c_T| \ll 1$ during inflation. The tensor spectrum and tensor-to-scalar ratio are

$$\mathcal{P}_T(k) = \frac{2H_*^2}{\pi^2 M_{{\rm eff},*}^2\, c_{T,*}}\,, \qquad r \equiv \frac{\mathcal{P}_T}{\overline{\mathcal{P}}_\zeta} \simeq 16\,\epsilon_{\eta,*}\,\frac{\overline{c}_{s,*}}{c_{T,*}}\,. \tag{3.51}$$

This yields an immediate inversion

$$\epsilon_{\eta,*} \simeq \frac{r}{16}\,\frac{c_{T,*}}{\overline{c}_{s,*}}\,, \qquad H_*^2 \simeq 8\pi^2\, M_{{\rm eff},*}^2\, \epsilon_{\eta,*}\, \overline{c}_{s,*}\, \overline{\mathcal{P}}_\zeta\,, \qquad V_{\eta,*} \simeq 3M_{{\rm eff},*}^2 H_*^2, \tag{3.52}$$

so a measurement of $(\overline{\mathcal{P}}_\zeta, n_s, r)$ pins down $(H_*, \epsilon_{\eta,*})$ and the energy scale $V_{\eta,*}$ up to the mild $c_T/\overline{c}_s$ ratio.

3.5.3 Controlled statistical anisotropy

Let the sound–speed anisotropy be small: $c_\parallel^2 = c_0^2 + \delta c^2$, $c_\perp^2 = c_0^2$, with $|\delta c^2| \ll c_0^2$. Expanding $c_s(\theta)$ to first order,

$$c_s(\theta) \simeq c_0 \left[1 + \frac{\delta c^2}{2c_0^2}\cos^2\theta\right], \qquad \mathcal{P}_\zeta(k,\theta) \simeq \overline{\mathcal{P}}_\zeta(k)\left[1 + g_*(k)\,\cos^2\theta\right],$$

with the quadrupolar amplitude

$$g_*(k) \;\simeq\; -\frac{\delta c^2}{2c_0^2} \;+\; \mathcal{O}\!\left(\frac{\delta c^2}{c_0^2}\right)^2. \tag{3.53}$$

Its scale dependence is $d\ln|g_*|/d\ln k \simeq -\dot{(\delta c^2/c_0^2)}/H$. CMB bounds at the percent level enforce the *anisotropy gate* $|\delta c^2|/c_0^2 \ll 10^{-2}$ during the observable e–folds, mapping to limits on director–sector mixings and background shear (Sec. **??**).

3.5.4 Non–Gaussianity: magnitude and shapes

Cubic interactions arise from the (η,ζ) sector and higher–derivative spectral curvature terms. Angle–averaged, the leading equilateral–type amplitude follows the single–clock logic with the effective sound speed:

$$f_{\rm NL}^{\rm equil} \;\sim\; \mathcal{O}\!\left(1 - \overline{c_s}^{-2}\right) \;+\; \mathcal{O}(\epsilon_\eta,\eta_\eta)\,, \qquad f_{\rm NL}^{\rm local} \;=\; \mathcal{O}(\epsilon_\eta,\eta_\eta). \tag{3.54}$$

Anisotropy induces a small direction–dependent modulation $\delta f_{\rm NL} \propto (\delta c^2/c_0^2)\cos^2\theta$. Within the η–wave regime that satisfies the anisotropy gate and slow–roll, non–Gaussianity is typically $|f_{\rm NL}| \lesssim \mathcal{O}(1)$.

3.5.5 Pivot mapping and reheating dependence

The e–folds between CMB pivot exit and the end of inflation, N_*, depend on the reheating history controlled by Γ_η and the effective equation of state $w_{\rm reh}$. A convenient parametrization is

$$N_* \;\simeq\; N_0 + \frac{1}{4}\ln\frac{V_{\eta,*}}{M_{{\rm eff},*}^4} - \frac{1-3w_{\rm reh}}{12(1+w_{\rm reh})}\,\ln\frac{\rho_{\rm reh}}{\rho_{\rm end}} + \Delta_M, \tag{3.55}$$

where N_0 collects known late–time contributions (today's a_0H_0 and pivot k_*), $\rho_{\rm end}$ is the energy density at $\epsilon_\eta = 1$, $\rho_{\rm reh}$ follows from Sec. 3.3, and Δ_M captures mild drift in $M_{\rm eff}$. Equation (3.55) is the bridge between observables (n_s, r) and model parameters $(V_\eta, \kappa_\eta, M_{\rm eff}, \Gamma_\eta)$.

3.5.6 Minimal inversion from observables to spectral parameters

Given a dataset $\mathcal{D} = \{\overline{\mathcal{P}}_\zeta(k_*),\, n_s,\, \alpha_s,\, r,\, g_*\}$, a minimal, assumption–light inversion proceeds as:

$$\epsilon_{\eta,*} = \frac{r}{16}\,\frac{c_{T,*}}{\overline{c}_{s,*}}, \qquad H_* = \sqrt{8\pi^2\, M_{{\rm eff},*}^2\,\epsilon_{\eta,*}\,\overline{c}_{s,*}\,\overline{\mathcal{P}}_\zeta}, \qquad V_{\eta,*} = 3M_{{\rm eff},*}^2 H_*^2, \tag{3.56}$$

$$\eta_{\eta,*} \simeq -\Big(n_s - 1 + 2\epsilon_{\eta,*} + s_{c,*} + \sigma_{M,*}\Big), \tag{3.57}$$

$$\frac{V_\eta'}{V_\eta}\Big|_* \simeq \sqrt{\frac{2\,\epsilon_{\eta,*}}{M_{{\rm eff},*}^2\,\kappa_{\eta,*}}}\left(1 + \frac{\Gamma_{\eta,*}}{3H_*\kappa_{\eta,*}}\right), \qquad \frac{V_\eta''}{V_\eta}\Big|_* \simeq \frac{\eta_{\eta,*}}{M_{{\rm eff},*}^2\,\kappa_{\eta,*}}, \tag{3.58}$$

$$\frac{\delta c^2}{c_0^2}\Big|_* \simeq -2\,g_*, \qquad |g_*| \ll 10^{-2} \;\;\Rightarrow\;\; \text{anisotropy gate satisfied.} \tag{3.59}$$

These relations determine the local slope/curvature of V_η at the pivot, modulo $(\kappa_\eta, \Gamma_\eta, M_{\rm eff})$, which can be constrained with reheating observables (Sec. 3.3) and the running α_s.

3.5.7 Operational consistency gates (observational layer)

- **(R1) Single adiabatic mode:** post–reheating isocurvature below current limits, consistent with the open–system damping of Sec. 3.4.4.
- **(R2) Causality:** $c_{T,*} \leq 1$, $0 < \bar{c}_{s,*} \leq 1$; late–time $|c_T - 1|$ obeys multi–messenger constraints.
- **(R3) Small anisotropy:** $|g_*| \ll 10^{-2}$ at CMB scales, mapping to $|\delta c^2|/c_0^2 \ll 10^{-2}$.
- **(R4) Non–Gaussianity:** $|f_{\rm NL}^{\rm local}| \lesssim \mathcal{O}(1)$ and $|f_{\rm NL}^{\rm equil}| \lesssim \mathcal{O}(1 - \bar{c}_s^{-2})$.
- **(R5) Restoration:** $c_T \to 1$ and $c_s(\theta) \to$ isotropic after the η condensate decoheres (consistent with Sec. 3.4.4).

Summary. Equations (3.47)–(3.59) provide a compact, fitting–ready interface between the η–wave mechanism and observation: (n_s, r, α_s, g_*) map to $(\epsilon_{\eta,*}, H_*, V_\eta'/V_\eta, V_\eta''/V_\eta, \delta c^2/c_0^2)$ with transparent roles for $(\kappa_\eta, \Gamma_\eta, M_{\rm eff})$. The anisotropy gate forces director–induced deviations to be small but testable; the tensor sector inherits the tight causal bound on c_T. Together these predictions set the stage for model–class exploration in Sec. 3.2 and for data–driven calibration in later chapters.

Part II

Spectral Cosmology: From Emergent Inflation to Structure

Chapter 4

Primordial Perturbations: Scalars, Tensors, and Non-Gaussianity

4.1 Scalar Sector: Curvature Mode ζ with Direction–Dependent Speed $c_s(\theta)$

This section derives the quadratic action, canonical normalization, and power spectrum of the adiabatic curvature perturbation ζ in the η–wave inflationary background (Chap. **??**), when spectral anisotropy induces a *direction–dependent* sound speed $c_s(\theta)$. Our starting point is the emergent geometry $g^{\rm eff}_{\mu\nu}$ and aether frame u^μ of Volume I, with slow background evolution ($\epsilon_\eta, \eta_\eta, s_c, \sigma_M \ll 1$; see below).

4.1.1 Setup, gauge choice, and constraints

We work in the aether–comoving gauge, where the internal clock T defines the time slicing, and choose vanishing aether momentum flux for scalars. On constant–T hypersurfaces, we write the spatial metric as

$$h_{ij} = a^2(t)\, e^{2\zeta(t,\mathbf{x})}\, \delta_{ij} + \cdots ,$$

with $a(t)$ the isotropized scale factor built from u^μ (Vol. I, Ch. 2). Lapse and shift are non-dynamical and are eliminated by their constraint equations, yielding a single-clock description by ζ.[1]

Spectral anisotropy originates from the internal director $\hat{\chi}$, splitting spatial gradients into components parallel/perpendicular to $\hat{\chi}$:

$$\nabla_\parallel \equiv \hat{\chi}\cdot\nabla, \qquad \nabla_\perp^2 \equiv \nabla^2 - \nabla_\parallel^2, \qquad \cos\theta \equiv \widehat{\mathbf{k}}\cdot\hat{\chi}.$$

To leading order, this produces an *elliptic* principal symbol with two sound speeds, $c_\parallel$ and $c_\perp$ (Vol. I, Sec. 2.3), summarized as

$$c_s^2(\theta) \;=\; c_\parallel^2\, \cos^2\theta + c_\perp^2\, \sin^2\theta, \qquad 0 < c_\perp^2 \le c_\parallel^2 \le 1, \tag{4.1}$$

with small anisotropy $|c_\parallel^2 - c_\perp^2| \ll 1$ during the observable e–folds.

4.1.2 Quadratic action and canonical normalization

After integrating out the constraints, the quadratic action for ζ takes the universal single–clock form

$$S_\zeta^{(2)} \;=\; \frac{1}{2}\int dt\, d^3x\; a^3(t)\, \mathcal{Q}_\zeta \left[\dot{\zeta}^2 - \frac{c_s^2(\theta)}{a^2(t)}(\partial_i\zeta)^2\right], \qquad \mathcal{Q}_\zeta \;=\; \frac{2\, M_{\rm eff}^2(\rho)\, \epsilon_\eta}{c_s^2(\theta)}. \tag{4.2}$$

Here $M_{\rm eff}(\rho)$ is the state–dependent Planck scale (Vol. I, Ch. 2–3) and $\epsilon_\eta \equiv -\dot{H}/H^2$ with $H \equiv \frac{1}{3}\nabla_\mu u^\mu$. The *no–ghost* gate requires $\mathcal{Q}_\zeta > 0$, automatically satisfied for $M_{\rm eff}^2 > 0$, $\epsilon_\eta > 0$.

[1] All background coefficients below are functionals of the structural state $\mathcal{S} = \{\rho, \eta, \Phi, \chi^\mu, T\}$ evaluated on the homogeneous η–wave attractor.

Switch to conformal time $d\tau \equiv dt/a$; the action becomes

$$S_\zeta^{(2)} \;=\; \frac{1}{2}\int d\tau\, d^3x\; z^2(\theta)\big[(\zeta')^2 - c_s^2(\theta)\,(\partial_i\zeta)^2\big]\,, \qquad z^2(\theta) \;\equiv\; 2a^2\,\mathcal{Q}_\zeta = \frac{4a^2 M_{\rm eff}^2\,\epsilon_\eta}{c_s^2(\theta)}. \tag{4.3}$$

Define the canonical Mukhanov variable $v(\tau,\mathbf{x};\theta) \equiv z(\theta)\,\zeta(\tau,\mathbf{x})$. In Fourier space, each $\mathbf{k}$–mode (with fixed θ) obeys

$$v_k''(\theta) + \Big[c_s^2(\theta)\,k^2 - \frac{z''}{z}(\theta)\Big]\,v_k(\theta) = 0. \tag{4.4}$$

4.1.3 Slow–variation background and Hankel solutions

Under spectral slow–variation (Chap. **??**), $c_s(\theta)$ is nearly time–independent and $z''/z \simeq (\nu^2 - \frac{1}{4})/\tau^2$ with

$$\nu \;\simeq\; \frac{3}{2} \;+\; \epsilon_\eta \;+\; \frac{\eta_\eta}{2} \;+\; \frac{s_c}{2} \;+\; \frac{\sigma_M}{2}, \qquad s_c \equiv \frac{\dot{\bar{c}}_s}{H\,\bar{c}_s}, \quad \sigma_M \equiv \frac{\dot{M}_{\rm eff}^2}{H\,M_{\rm eff}^2}, \tag{4.5}$$

where $\bar{c}_s$ denotes an angle–average (defined below). Equation (4.4) admits Hankel solutions normalized to a Bunch–Davies–like vacuum at early times ($-c_s k\tau \gg 1$):

$$v_k(\theta) \;=\; \frac{\sqrt{\pi}}{2}\,e^{i(\nu+\frac{1}{2})\pi/2}\,\sqrt{-\tau}\,H_\nu^{(1)}\big(c_s(\theta)k\,|\tau|\big). \tag{4.6}$$

On super–sound–horizon scales ($|c_s k\tau| \ll 1$), $\zeta_k = v_k/z$ freezes to a constant.

4.1.4 Two–point function and angular power spectrum

The curvature two–point function is

$$\langle\zeta_{\mathbf{k}}\,\zeta_{\mathbf{k}'}\rangle \;=\; (2\pi)^3\,\delta^{(3)}(\mathbf{k}+\mathbf{k}')\,\mathcal{P}_\zeta(k,\theta), \qquad \mathcal{P}_\zeta(k,\theta) \;=\; \frac{H_*^2}{8\pi^2\,M_{{\rm eff},*}^2\,\epsilon_{\eta,*}\,c_s(\theta)_*}\,, \tag{4.7}$$

evaluated at freeze-out $k = a_* H_*/c_s(\theta)_*$. Angle–averaging over $\widehat{\mathbf{k}}$ gives

$$\overline{\mathcal{P}}_\zeta(k) \;\equiv\; \frac{1}{4\pi}\int d\Omega_{\hat{k}}\,\mathcal{P}_\zeta(k,\theta) \;=\; \frac{H_*^2}{8\pi^2\,M_{{\rm eff},*}^2\,\epsilon_{\eta,*}\,\bar{c}_s}\,, \qquad \bar{c}_s^{\,-1} \;\equiv\; \frac{1}{2}\int_0^\pi d\theta\,\sin\theta\,\frac{1}{c_s(\theta)}. \tag{4.8}$$

For small anisotropy, write $c_\parallel^2 = c_0^2 + \delta c^2$, $c_\perp^2 = c_0^2$ with $|\delta c^2| \ll c_0^2$. Then

$$\mathcal{P}_\zeta(k,\theta) \;=\; \overline{\mathcal{P}}_\zeta(k)\,\big[1 + g_*(k)\cos^2\theta\big] \;+\; \mathcal{O}\big((\delta c^2/c_0^2)^2\big), \qquad g_*(k) \simeq -\frac{\delta c^2}{2c_0^2} \;+\; \mathcal{O}\big(\epsilon_\eta\big). \tag{4.9}$$

CMB bounds enforce $|g_*| \ll 10^{-2}$ on observable scales (Chap. **??**).

4.1.5 Tilt, running, and horizon–crossing conditions

The angle–averaged tilt and running follow from slow variation of background quantities at horizon exit:

$$n_s - 1 \equiv \frac{d\ln\overline{\mathcal{P}}_\zeta}{d\ln k} \simeq -2\epsilon_\eta - \eta_\eta - s_c - \sigma_M, \tag{4.10}$$

$$\alpha_s \equiv \frac{dn_s}{d\ln k} \simeq -2\epsilon_\eta\eta_\eta - \xi_\eta^2 - \frac{\dot{s}_c}{H} - \frac{\dot{\sigma}_M}{H}, \qquad \xi_\eta^2 \equiv \frac{M_{\rm eff}^4\,\kappa_\eta^2\,V_\eta' V_\eta'''}{V_\eta^2}. \tag{4.11}$$

The freeze–out condition is *directional*:

$$k_*(\theta) \;=\; \frac{a_* H_*}{c_s(\theta)_*}, \qquad \text{or equivalently} \quad |c_s(\theta)k\tau|_* \simeq 1, \tag{4.12}$$

which slightly shifts the effective e–fold $N_*(\theta)$ at which a given k exits. This dependence averages out in $\overline{\mathcal{P}}_\zeta(k)$ up to $\mathcal{O}(g_*)$ corrections.

4.1.6 Validity, stability, and restoration

The scalar description remains valid provided:

- **No ghosts/hyperbolicity:** $\mathcal{Q}_\zeta > 0$ and $0 < c_\perp^2 \le c_\parallel^2 \le 1$.
- **Weak anisotropy:** $|c_\parallel^2 - c_\perp^2| \ll 1$ during the observable band (ensures $|g_*| \ll 10^{-2}$).
- **Single clock:** entropy modes are heavy or damped so that ζ is conserved on super–horizon scales.
- **Restoration:** after reheating (Sec. 3.3) the effective propagation becomes isotropic and $c_T \to 1$, recovering standard adiabatic evolution (cf. Sec. 4.4).

Summary. The curvature mode in the η–wave background is governed by the quadratic action (4.2) with a directional sound speed (4.1). Quantization yields an angle–dependent spectrum (4.7) and a controlled, quadrupolar statistical anisotropy (4.9). Angle–averaged observables $(\overline{\mathcal{P}}_\zeta, n_s, \alpha_s)$ reduce to familiar single–clock expressions with spectral corrections (s_c, σ_M) that encode the slow drift of c_s and $M_{\rm eff}$.

4.2 Tensor Sector: $c_T \simeq 1 + \delta c_T(\mathcal{S})$

Tensor perturbations are the transverse–traceless (TT) excitations of the emergent metric $g_{\mu\nu}^{\rm eff}$ around the η–wave background. They probe the curvature sector of the unified action (Vol. I, Ch. 3) and, in our framework, propagate with a speed *close* to luminal,

$$c_T^2 \;=\; 1 + \delta c_T(\mathcal{S}), \qquad |\delta c_T| \ll 1, \tag{4.13}$$

where δc_T is a small, state–dependent functional of the structural fields $\mathcal{S} = \{\rho, \eta, \Phi, \chi^\mu, T\}$. We allow mild time–dependence (slow variation) and—optionally—small anisotropy inherited from the internal director $\hat{\chi}$; both are bounded by observational gates discussed below.

4.2.1 Quadratic action, canonical normalization, and equation of motion

Let h_{ij} denote the TT metric perturbation on constant–T hypersurfaces: $h^i{}_i = 0$, $\partial_i h^{ij} = 0$. To quadratic order,

$$S_T^{(2)} \;=\; \frac{M_{\rm eff}^2(\rho)}{8} \int d^4x\, a^3(t) \left[\dot{h}_{ij}\dot{h}_{ij} - \frac{c_T^2}{a^2}(\partial_k h_{ij})(\partial_k h_{ij}) - m_T^2(\mathcal{S})\, h_{ij}h_{ij} \right], \tag{4.14}$$

with $m_T^2(\mathcal{S})$ a (typically negligible) effective mass generated by higher–curvature operators or transient anisotropies; we assume $|m_T^2| \ll H^2$ during the observable window. In conformal time $d\tau = dt/a$, expand in polarizations,

$$h_{ij}(\tau, \mathbf{x}) = \sum_{\lambda=+,\times} \int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, e_{ij}^{(\lambda)}(\hat{\mathbf{k}})\, h_\lambda(\tau, \mathbf{k}), \quad e_{ii}^{(\lambda)} = 0, \quad k_i e_{ij}^{(\lambda)} = 0, \quad e_{ij}^{(\lambda)} e_{ij}^{(\lambda')} = 2\delta_{\lambda\lambda'}.$$

Define the canonical variables

$$v_\lambda(\tau, \mathbf{k}) \;\equiv\; z_T\, h_\lambda(\tau, \mathbf{k}), \qquad z_T(\tau) \;\equiv\; \frac{a(\tau)\, M_{\rm eff}(\rho)}{2}, \tag{4.15}$$

so that each mode obeys

$$v_\lambda'' + \left[c_T^2(\theta)\, k^2 - \frac{z_T''}{z_T} + a^2 m_T^2 \right] v_\lambda \;=\; 0, \qquad \frac{z_T''}{z_T} \;\simeq\; \frac{\nu_T^2 - \frac{1}{4}}{\tau^2}. \tag{4.16}$$

Slow variation of $M_{\rm eff}$ and c_T implies

$$\nu_T \;\simeq\; \frac{3}{2} \;+\; \epsilon_\eta \;+\; \frac{\sigma_M}{2} \;+\; \frac{s_T}{2}, \qquad \sigma_M \equiv \frac{\dot{M}_{\rm eff}^2}{H M_{\rm eff}^2}, \quad s_T \equiv \frac{\dot{c}_T}{H c_T}, \tag{4.17}$$

with $\epsilon_\eta \equiv -\dot{H}/H^2$ as in Sec. 4.1. A Bunch–Davies–like vacuum at early times ($-c_T k\tau \gg 1$) gives the Hankel solution

$$v_\lambda(\tau, k) \;=\; \frac{\sqrt{\pi}}{2}\, e^{i(\nu_T+\frac{1}{2})\pi/2}\, \sqrt{-\tau}\, H^{(1)}_{\nu_T}\big(c_T k\,|\tau|\big), \tag{4.18}$$

and *tensor freeze–out* occurs at

$$k_*(\theta) \;=\; \frac{a_* H_*}{c_T(\theta)_*}\,, \tag{4.19}$$

with a possible weak dependence on the angle $\theta = \angle(\hat{\mathbf{k}}, \hat{\chi})$ if c_T is anisotropic.

4.2.2 Tensor spectrum, tilt, and tensor–to–scalar ratio

At leading order (neglecting m_T^2 and to first order in slow variation), the dimensionless tensor power per polarization is

$$\mathcal{P}_T^{(\lambda)}(k,\theta) \;=\; \frac{H_*^2}{\pi^2\, M_{\mathrm{eff},*}^2\, c_T(\theta)_*}\,, \qquad \mathcal{P}_T \;\equiv\; \sum_\lambda \mathcal{P}_T^{(\lambda)} \;=\; \frac{2H_*^2}{\pi^2\, M_{\mathrm{eff},*}^2\, c_T(\theta)_*}\,. \tag{4.20}$$

The tensor tilt follows from the slow drift of $(H, M_{\mathrm{eff}}, c_T)$ at horizon crossing,

$$n_T \;\equiv\; \frac{d\ln \mathcal{P}_T}{d\ln k} \;\simeq\; -2\epsilon_\eta \;-\; \sigma_M \;-\; s_T, \tag{4.21}$$

reducing to $n_T \simeq -2\epsilon_\eta$ when M_{eff} and c_T are constant. Using the scalar spectrum (Sec. 4.1), the tensor–to–scalar ratio is

$$r(k) \;\equiv\; \frac{\mathcal{P}_T}{\mathcal{P}_\zeta} \;\simeq\; 16\,\epsilon_\eta\, \frac{\bar{c}_s}{c_T}\, \left[1 + \mathcal{O}(g_*)\right], \tag{4.22}$$

with $\bar{c}_s$ the angular average of $c_s(\theta)$ (Eq. (4.8)) and g_* the small quadrupolar anisotropy parameter of Eq. (4.9). In the isotropic limit ($c_T = \bar{c}_s = 1$), $r = 16\epsilon_\eta$.

4.2.3 Anisotropy, birefringence, and parity violation (optional effects)

The internal director $\hat{\chi}$ can induce a suppressed *directional* correction $c_T(\theta) = c_{T,0}\left[1 + \delta_T \cos^2\theta + \cdots\right]$ with $|\delta_T| \ll 1$, leading to an angular modulation of $\mathcal{P}_T$ analogous to Eq. (4.9). In addition, higher–order topological terms (e.g. Chern–Simons/WZW–like densities in the emergent curvature sector, cf. Vol. I, App. 2.10) may produce tiny *parity–violating* birefringence, splitting the two polarizations:

$$\mathcal{P}_T^{(+)} - \mathcal{P}_T^{(\times)} \;\propto\; \xi_{\mathrm{CS}}\, \frac{H_*}{M_{\mathrm{eff},*}} \;+\; \mathcal{O}(\xi_{\mathrm{CS}}^2), \qquad |\xi_{\mathrm{CS}}| \ll 1, \tag{4.23}$$

which we treat as a controlled higher–order effect subject to stringent bounds.

4.2.4 Consistency gates and observational bounds

During the inflationary window and subsequently:

- **(T1) No ghosts/hyperbolicity:** $M_{\mathrm{eff}}^2 > 0$, $0 < c_T^2 \le 1$.
- **(T2) Small drift:** $|s_T|, |\sigma_M| \ll 1$ so that n_T respects slow–roll ordering.
- **(T3) Anisotropy gate:** $|\delta_T| \ll 1$; any θ–dependence of c_T must be below bounds from CMB B–modes and large–scale structure.
- **(T4) Restoration:** post–inflation, $c_T \to 1$ rapidly to satisfy late–time multi–messenger constraints and ensure GR–like propagation in the current epoch.
- **(T5) Masslessness:** $|m_T^2| \ll H^2$ during horizon exit to preserve the nearly scale–invariant tensor spectrum.

Summary. Gravitational waves in the Quantum–Aether cosmology propagate with a near–luminal speed $c_T \simeq 1+\delta c_T(\mathcal{S})$, yielding the tensor spectrum (4.20), tilt (4.21), and the generalized consistency relation (4.22). Mild anisotropy and parity–violating effects are possible but tightly constrained; the restoration gate enforces $c_T \to 1$ after reheating, ensuring compatibility with late–time observations while retaining distinctive inflationary–era signatures.

4.3 Statistical Anisotropy and Shape of Non–Gaussianity

The internal director $\hat{\chi}$ and the mild anisotropy of the principal symbol (Vol. I, Ch. 2) can leave controlled, *direction–dependent* imprints on the primordial spectra. In this section we derive (i) the quadrupolar modulation of the scalar power spectrum, (ii) the leading bispectrum templates and their angular dependences, (iii) generalized single–clock consistency relations in the squeezed limit, and (iv) the expected size of the trispectrum in the collapsed limit. Throughout we assume that the anisotropy is small and slowly varying, consistent with the inflationary gates of Sec. 4.4.

4.3.1 Sources of statistical anisotropy

Two primary sources generate statistical anisotropy in the scalar sector:

1. **Anisotropic sound speed.** The director–dependent propagation in the scalar channel (Sec. 4.1) implies
$$c_s(\theta) \;=\; \bar{c}_s\big[1+\delta_s\,\cos^2\theta+\mathcal{O}(\delta_s^2)\big]\,, \qquad |\delta_s|\ll 1, \quad \cos\theta\equiv\hat{\mathbf{k}}\cdot\hat{\chi}, \tag{4.24}$$
with $\bar{c}_s$ the angle–averaged sound speed. This dependence feeds into $\mathcal{P}_\zeta \propto (H^2)/(M_{\rm eff}^2\,\epsilon_\eta\,c_s)$.
2. **Weak shear or director dynamics.** A small background shear $\sigma/H \ll 1$ and slow drift of $\hat{\chi}$ generate subleading modulations that can be captured by higher–derivative operators in the effective action; we keep these at the level of order–of–magnitude estimates.

4.3.2 Quadrupolar modulation of the power spectrum

Using Eq. (4.8) for the angle–averaged spectrum and expanding to first order in δ_s, we obtain

$$\mathcal{P}_\zeta(k,\theta) = \frac{H_*^2}{8\pi^2 M_{{\rm eff},*}^2\,\epsilon_{\eta,*}\,c_s(\theta)_*} \;\simeq\; \overline{\mathcal{P}}_\zeta(k)\,\Big[1+g_*(k)\,P_2(\cos\theta)\Big], \tag{4.25}$$
$$g_*(k)\equiv -\frac{2}{3}\,\delta_s(k) \;+\; \mathcal{O}\!\left(\frac{\sigma^2}{H^2}\right), \qquad P_2(\mu)=\tfrac{1}{2}(3\mu^2-1),$$

where the numerical coefficient arises from projecting the $\cos^2\theta$ dependence onto the Y_{2m} basis.[2] The *anisotropy parameter* $g_*(k)$ thus directly probes the small departure of c_s from isotropy. The slow variation of background quantities implies a weak running

$$\frac{d\ln|g_*|}{d\ln k} \;\simeq\; -\,s_c \;+\; \mathcal{O}\!\left(\frac{\dot{\delta}_s}{H\delta_s}\right), \qquad s_c\equiv\big\langle \dot{c}_s/(Hc_s)\big\rangle_\theta. \tag{4.26}$$

Gate: CMB isotropy bounds require $|g_*|\ll 10^{-2}$ during horizon exit of observable modes; we enforce this as a prior on δ_s and shear contributions.

4.3.3 Cubic action and anisotropic bispectrum

Expanding the unified action to cubic order in ζ (and keeping the leading operators after integrating out nondynamical fields) yields, in comoving–aether gauge,

$$S_\zeta^{(3)} = \int d^4x\,a^3\Big\{A_1\,\zeta\,\dot{\zeta}^2 \;+\; A_2\,\zeta\,\frac{(\partial_i\zeta)^2}{a^2} \;+\; A_3\,\dot{\zeta}\,\frac{\hat{\chi}^i\hat{\chi}^j\,\partial_i\zeta\,\partial_j\zeta}{a^2} \;+\; A_4\,\zeta\,\frac{\hat{\chi}^i\hat{\chi}^j\,\partial_i\partial_j\zeta}{a^2} \;+\;\cdots\Big\}, \tag{4.27}$$

[2] One may equivalently write $\mathcal{P}_\zeta(\mathbf{k}) = \overline{\mathcal{P}}_\zeta(k)\big[1+g_*(\hat{\mathbf{k}}\cdot\hat{\chi})^2\big]$ up to a constant redefinition; Eq. (4.25) uses the conventional P_2 form.

where $A_{1,2}$ reduce to the familiar single–clock coefficients in the isotropic limit and $A_{3,4} \propto \delta_s$ encode the leading director–induced anisotropies.[3] The three–point function $\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle = (2\pi)^3 \delta^{(3)}(\sum \mathbf{k}_i)\, B_\zeta(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ can be written as

$$B_\zeta(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \;=\; \sum_X f_{\rm NL}^{(X)}\, \mathcal{S}_X(k_1, k_2, k_3) \Big[1 + \alpha_X\, \mathcal{A}_2(\hat{\mathbf{k}}_i \cdot \hat{\chi})\Big], \tag{4.28}$$

where $X \in \{\text{local}, \text{equil}, \text{ortho}\}$ are standard momentum templates, $\mathcal{S}_X$ their shape functions (normalized in the isotropic case), and $\mathcal{A}_2$ the quadrupolar angular polynomial built from $P_2(\hat{\mathbf{k}}_a \cdot \hat{\chi})$. A representative estimate gives

$$f_{\rm NL}^{\rm eff} \;\sim\; \underbrace{\mathcal{O}\!\left(\frac{1-\bar{c}_s^2}{\bar{c}_s^2}\right)}_{\text{single–clock (equilateral–like)}} \;+\; \underbrace{\mathcal{O}(\delta_s)}_{\text{anisotropic mixing}}, \qquad \alpha_X \;\sim\; \mathcal{O}(\delta_s), \tag{4.29}$$

with α_X differing slightly among templates due to the tensor structure of $A_{3,4}$.

4.3.4 Squeezed limit and generalized consistency

In single–clock, isotropic inflation one has Maldacena's relation $f_{\rm NL}^{\rm local}(k_L \ll k_S) = \frac{5}{12}(1 - n_s)$. Here the long–wavelength mode modulates both the local clock and, at $\mathcal{O}(\delta_s)$, the *directional* sound speed. A background–wave argument gives

$$\lim_{k_L \ll k_S} B_\zeta \;=\; -\left[\frac{d \ln \mathcal{P}_\zeta(k_S, \theta_S)}{d \ln k_S}\right] \mathcal{P}_\zeta(k_L)\, \mathcal{P}_\zeta(k_S, \theta_S) \;+\; \mathcal{O}\big(\partial_{\theta_S} \ln \mathcal{P}_\zeta\big), \tag{4.30}$$

hence

$$f_{\rm NL}^{\rm squeezed}(\theta_S) \;\simeq\; \frac{5}{12}\Big[(1 - n_s) \;+\; \Delta_{\rm aniso}\, P_2(\cos\theta_S)\Big], \qquad \Delta_{\rm aniso} \;\propto\; \delta_s \;+\; \mathcal{O}\!\left(\frac{\sigma^2}{H^2}\right). \tag{4.31}$$

Thus the squeezed bispectrum acquires a small quadrupolar modulation aligned with $\hat{\chi}$, controlled by the same parameter that sets g_*.

4.3.5 Trispectrum and collapsed limit

Quartic interactions and exchange of an anisotropic scalar generate a trispectrum $\langle \zeta^4 \rangle_c = (2\pi)^3 \delta^{(3)}(\sum \mathbf{k}_i)\, T_\zeta$ with amplitudes $(\tau_{\rm NL}, g_{\rm NL})$. Parametrically,

$$\tau_{\rm NL} \;\sim\; \mathcal{O}\!\left(\frac{1}{\bar{c}_s^4}\right) \;+\; \mathcal{O}\!\left(\frac{\delta_s^2}{\bar{c}_s^2}\right), \qquad g_{\rm NL} \;\sim\; \mathcal{O}\!\left(\frac{1}{\bar{c}_s^4}\right) \;+\; \mathcal{O}(\delta_s^2), \tag{4.32}$$

and the collapsed limit inherits a quadrupolar angular dependence analogous to Eq. (4.28). Inclusive soft theorems (Vol. I, Sec. 6.15) ensure IR finiteness when radiative corrections are included.

4.3.6 Consistency gates and observational priors

- **(A1) Small anisotropy:** $|\delta_s| \ll 1$ so that $|g_*| \ll 10^{-2}$ and angular modulations remain perturbative.
- **(A2) Sub–luminality & stability:** $0 < c_{\perp,\zeta}^2 \le c_{\parallel,\zeta}^2 \le 1$; cubic coefficients obey positivity bounds (no gradient instabilities).
- **(A3) Single–clock restoration:** In the limit $\delta_s \to 0$ the standard relations are recovered: $g_* \to 0$, $f_{\rm NL}^{\rm squeezed} \to \frac{5}{12}(1 - n_s)$.
- **(A4) Radiative control:** Loop–induced anisotropies from integrating out heavy modes are $\mathcal{O}(\delta_s\, \epsilon_\eta)$ or smaller (technical naturalness).
- **(A5) Observational priors:** Imposed bounds on $(g_*, f_{\rm NL}, \tau_{\rm NL})$ from CMB/LSS analyses serve as filters on the parameter set $(\bar{c}_s, \delta_s, \epsilon_\eta, \dots)$ in the spectral EFT.

[3] Additional operators involving $\delta\chi$ mixing can be included; at leading order they renormalize $A_{3,4}$ after integrating out heavy anisotropic modes.

Summary. Statistical anisotropy in the Quantum–Aether cosmology is a controlled, calculable consequence of the director–induced modulation of the scalar sound speed. It generates a quadrupolar power–spectrum modulation $g_*(k) \propto -\delta_s(k)$, small angular deformations of the bispectrum and trispectrum, and a generalized squeezed–limit relation. All effects are aligned with $\hat{\chi}$ and bounded by the same small parameter δ_s, providing a sharp, multifrequency target for observational tests.

4.4 Consistency Gates for the Primordial Sector

We collect the *non–negotiable* mathematical conditions that render the primordial sector (scalars and tensors with mild statistical anisotropy) causal, stable, radiatively controlled, and continuously connected to the post–inflationary adiabatic regime. Each gate is stated as an inequality or operator–positivity requirement, followed by its diagnostic form in terms of the parameters introduced in Secs. 4.1–4.3.

G1. No–ghost gate (time–kinetic positivity)

For the scalar curvature mode ζ with quadratic action

$$S_\zeta^{(2)} = \frac{1}{2}\int d^4x\, a^3\, \mathcal{Q}_\zeta \Big[\dot{\zeta}^2 - c_s^2(\theta)\, a^{-2}(\partial_i\zeta)^2\Big], \qquad \mathcal{Q}_\zeta = \frac{2M_{\rm eff}^2\,\epsilon_\eta}{c_s^2(\theta)}\,,$$

the absence of ghosts requires

$$\mathcal{Q}_\zeta > 0 \quad\Longleftrightarrow\quad M_{\rm eff}^2 > 0,\;\; \epsilon_\eta > 0,\;\; c_s^2(\theta) > 0 \;\text{ for all } \theta. \tag{4.33}$$

For the tensor sector

$$S_T^{(2)} = \frac{M_{\rm eff}^2}{8}\int d^4x\, a^3 \Big[\dot{h}_{ij}^2 - c_T^2\, a^{-2}(\partial_k h_{ij})^2\Big],$$

one needs $M_{\rm eff}^2 > 0$.

G2. Hyperbolicity, subluminality, and gradient stability

Let the principal symbol in the aether frame be

$$\omega^2 = c_s^2(\theta)\, k^2 + \cdots\,, \qquad \omega_T^2 = c_T^2\, k^2 + \cdots\,.$$

Then

$$0 < c_s^2(\theta) \le 1, \qquad 0 < c_T^2 \le 1, \qquad \forall\,\theta \in [0,\pi], \tag{4.34}$$

ensures well–posedness (hyperbolicity) and no superluminal group velocities in the $g_{\rm eff}$ cone. Define the anisotropy amplitude

$$\Delta_s \equiv \max_\theta \frac{|c_s^2(\theta) - \bar{c}_s^2|}{\bar{c}_s^2}\,,$$

and impose $\Delta_s \ll 1$ (CMB isotropy prior).

G3. Statistical–anisotropy bound (power spectrum)

From Sec. 4.3, the quadrupolar modulation is

$$\mathcal{P}_\zeta(k,\theta) = \overline{\mathcal{P}}_\zeta(k)\Big[1 + g_*(k)\, P_2(\cos\theta)\Big], \qquad g_*(k) \simeq -\tfrac{2}{3}\,\delta_s(k).$$

We require

$$|g_*(k)| \;=\; \frac{2}{3}\,|\delta_s(k)| \;\ll\; 10^{-2} \quad \text{on CMB/LSS scales}, \tag{4.35}$$

with slow running $|d\ln g_*/d\ln k| \ll 1$.

G4. Squeezed–limit consistency (single–clock restoration)

The generalized consistency relation (Sec. 4.3.4) reads

$$f_{\rm NL}^{\rm squeezed}(\theta_S) \simeq \frac{5}{12}\big[(1-n_s) + \Delta_{\rm aniso}\, P_2(\cos\theta_S)\big],$$

with $\Delta_{\rm aniso} \propto \delta_s$. The *restoration* gate demands

$$\lim_{\delta_s\to 0} f_{\rm NL}^{\rm squeezed} = \frac{5}{12}(1-n_s)\,, \qquad \lim_{\delta_s\to 0} g_* = 0\,, \tag{4.36}$$

ensuring the standard single–clock limit.

G5. Radiative control and technical naturalness

Loop corrections renormalize (c_s, δ_s, c_T) by amounts suppressed by slow parameters and the EFT cutoff $\Lambda_{\rm spec}$:

$$\delta c_s^{\rm loop},\ \delta(\delta_s)^{\rm loop},\ \delta c_T^{\rm loop} = \mathcal{O}\Big(\epsilon_\eta\, \frac{H^2}{\Lambda_{\rm spec}^2}\Big) \ +\ \mathcal{O}(\delta_s\, \epsilon_\eta)\,. \tag{4.37}$$

We require stability under RG flow across the observable window: no large logs or operator mixings that would violate (4.34)–(4.36).

G6. Backreaction and energy budget

The energy in fluctuations must remain subdominant:

$$\langle\delta\rho\rangle \ \sim\ \int^{k_{\rm max}} \frac{d^3k}{(2\pi)^3}\Big[\mathcal{Q}_\zeta\, \omega_k^2\, |\zeta_k|^2\Big] \ \ll\ 3M_{\rm eff}^2 H^2, \tag{4.38}$$

with $k_{\rm max}$ the EFT validity scale. A similar bound holds for tensors.

G7. Tensor gate (speed, normalization, and mixing)

The tensor action yields

$$\mathcal{P}_T = \frac{2H^2}{\pi^2 M_{\rm eff}^2 c_T}, \qquad r \simeq 16\,\epsilon_\eta\, \frac{\bar{c}_s}{c_T}.$$

We impose

$$|c_T - 1| \ll 1, \qquad c_T > 0, \qquad \text{no scalar–tensor kinetic mixing at leading order}, \tag{4.39}$$

and small anisotropic stress sources from the structural sector.

G8. Post–inflationary adiabaticity and initial conditions

After the η–wave reheats (Sec. 3.3), non–adiabatic pressure sources satisfy

$$\frac{\delta p_{\rm nad}}{\rho + p} \ \to\ 0 \quad\Longrightarrow\quad \dot{\zeta} \to 0 \ \text{ on } k \ll aH, \tag{4.40}$$

delivering adiabatic initial conditions for the Boltzmann hierarchy and suppressing iso–curvature.

G9. Inclusive IR safety (soft theorems)

Soft emission and long–mode dressing obey the inclusive finiteness conditions (cf. Vol. I, Sec. 6.15), so that loop–corrected two– and three–point functions are IR–finite after properly summing degenerate configurations. This protects Eqs. (4.35)–(4.36) from IR pathologies.

Operational checklist (diagnostics)

For any parameter choice $\Pi = \{M_{\rm eff}, \epsilon_\eta, \bar{c}_s, \delta_s, c_T, \dots\}$:

1. Compute $\mathcal{Q}_\zeta,\ c_s^2(\theta),\ c_T^2$ and verify (4.33)–(4.34).
2. Extract $g_*(k)$ from $c_s(\theta)$ via (4.25); check (4.35).
3. Evaluate $n_s - 1$ and $f_{\rm NL}^{\rm squeezed}(\theta)$; enforce (4.36).
4. Estimate loop shifts using (4.37) with $\Lambda_{\rm spec}$; require smallness.
5. Integrate fluctuation energy (4.38) up to $k_{\rm max}$; ensure subdominance.
6. Check tensor gate (4.39) and compute r consistency with the chosen ϵ_η.
7. Verify adiabatic restoration (4.40) in explicit reheating solutions.

Summary. Gates **G1**–**G9** carve out a mathematically consistent domain where the primordial sector is ghost–free, hyperbolic, weakly anisotropic, radiatively stable, and smoothly matches adiabatic late–time physics. They provide both analytic inequalities and a practical pipeline for numerical scans used in Chap. **??**.

Chapter 5

Defect Formation and Dark Solitons

5.1 Homotopy, Kibble–Zurek Mechanism, and Spectral Anisotropy

Order parameter manifold and topological classes

In the Quantum–Aether framework (Vol. I), the long-wavelength *order parameter* is a composite map

$$\mathcal{O}:\ \mathbb{R}^3 \ \longrightarrow\ \mathcal{M}_{\rm ord}\ \equiv\ \frac{\mathcal{F}_{\rm spec}}{\mathcal{H}}\,, \qquad \mathcal{F}_{\rm spec} = \text{internal spectral fiber built from } (\Phi,\rho,\eta,\chi^\mu,T)\,, \tag{5.1}$$

where the denominator $\mathcal{H}\subset\mathcal{F}_{\rm spec}$ is the *residual* symmetry subgroup preserved by the broken phase. The topology of $\mathcal{M}_{\rm ord}$ governs the admissible defect classes via homotopy:

$$\pi_0(\mathcal{M}_{\rm ord})\neq 0\ \Rightarrow\ \text{domain walls}\,, \tag{5.2}$$
$$\pi_1(\mathcal{M}_{\rm ord})\neq 0\ \Rightarrow\ \text{strings/vortices}\,, \tag{5.3}$$
$$\pi_2(\mathcal{M}_{\rm ord})\neq 0\ \Rightarrow\ \text{monopoles/hedgehogs}\,, \tag{5.4}$$
$$\pi_3(\mathcal{M}_{\rm ord})\neq 0\ \Rightarrow\ \text{Skyrmions/Hopf textures}\,. \tag{5.5}$$

Concrete realizations in our model include:

- **Walls** from discrete vacua in the phase sector of Φ (e.g. spontaneous breakdown of a $\mathbb{Z}_2$ remnant within a multi-branch potential for $\alpha=\arg\Phi$), hence $\pi_0\neq 0$.
- **Strings** from a $U(1)$ phase order parameter α with $\pi_1(U(1))=\mathbb{Z}$, realized by spectral-phase windings supported by the compensating response of χ^μ (Vol. I, Ch. 4).
- **Monopoles** from a unit vector $n^A(\Phi,\chi)\in S^2$ (e.g. normalized triplet extracted from $(\mathrm{Re}\,\Phi,\mathrm{Im}\,\Phi,\chi\cdot u)$), with $\pi_2(S^2)=\mathbb{Z}$.
- **Hopf textures** from maps $S^3\to S^2$ built out of (Φ,χ) where space is compactified ($\mathbb{R}^3\cup\{\infty\}\simeq S^3$), carrying Hopf invariant

$$Q_H\ =\ \frac{1}{32\pi^2}\int_{\mathbb{R}^3}\epsilon^{ijk}\,A_i\,F_{jk}\,d^3x, \qquad F_{ij}=\partial_iA_j-\partial_jA_i,\ \ A \text{ a preimage connection for } n\,. \tag{5.6}$$

Spectral metric and director. The emergent geometry $g^{\rm eff}_{\mu\nu}[\mathcal{S}]$ and the internal director $\hat\chi$ break spatial isotropy at the level of the *principal symbol*. In the aether rest frame u^μ, we decompose

$$\mathbf{k}=k_\parallel\,\hat\chi^{(0)}+\mathbf{k}_\perp, \qquad k_\perp^2\equiv\mathbf{k}\cdot\mathbf{k}-k_\parallel^2\,, \tag{5.7}$$

and the linearized growth/dispersion (cf. Vol. II, Sec. **??**) reads

$$\lambda_{\max}(\mathbf{k})\ =\ \lambda_0\ -\ D_\parallel\,k_\parallel^2\ -\ D_\perp\,k_\perp^2\ +\ \mathcal{O}(k^4), \qquad D_\parallel\neq D_\perp\ \text{ in general.} \tag{5.8}$$

This anisotropy feeds directly into the freeze-out scales of the Kibble–Zurek mechanism.

From modulational onset to topological freezing

Seed scale from linear instability. If the onset $\lambda_0 > 0$ (spectral modulational instability), the fastest-growing band is peaked near

$$k_{\text{peak},\parallel} \sim 0, \qquad k_{\text{peak},\perp} \sim \sqrt{\lambda_0/(2D_\perp)} \quad (\text{for } D_\perp < D_\parallel), \tag{5.9}$$

with an intrinsic *seed* length $\xi_{\text{seed}} \sim k_{\text{peak}}^{-1}$. This sets the earliest imprint for pattern scales prior to nonlinearity and defect formation.

Kibble–Zurek with spectral anisotropy. Across a continuous transition driven by a control parameter $\varepsilon(t)$ (e.g. an effective mass or coupling extracted from $\mathcal{S}$) swept at rate $\dot{\varepsilon} \approx 1/\tau_Q$, the equilibrium correlation lengths and relaxation times obey

$$\xi_\parallel(\varepsilon) \sim \xi_{0,\parallel}\,|\varepsilon|^{-\nu_\parallel}, \quad \xi_\perp(\varepsilon) \sim \xi_{0,\perp}\,|\varepsilon|^{-\nu_\perp}, \qquad \tau_{\parallel,\perp}(\varepsilon) \sim \tau_{0,\parallel/\perp}\,|\varepsilon|^{-z_{\parallel/\perp}\nu_{\parallel/\perp}}. \tag{5.10}$$

The *freeze-out* condition in each sector,

$$\tau_{\parallel,\perp}(\hat{\varepsilon}) \sim \left|\frac{\hat{\varepsilon}}{\dot{\varepsilon}}\right| \sim |\hat{\varepsilon}|\,\tau_Q, \tag{5.11}$$

gives the anisotropic Kibble–Zurek scales (for a linear quench $\varepsilon = t/\tau_Q$):

$$|\hat{\varepsilon}|_\parallel \sim \left(\frac{\tau_{0,\parallel}}{\tau_Q}\right)^{\frac{1}{1+z_\parallel\nu_\parallel}}, \qquad \hat{\xi}_\parallel \equiv \xi_\parallel(\hat{\varepsilon}_\parallel) \sim \xi_{0,\parallel}\left(\frac{\tau_Q}{\tau_{0,\parallel}}\right)^{\frac{\nu_\parallel}{1+z_\parallel\nu_\parallel}}, \tag{5.12}$$

$$|\hat{\varepsilon}|_\perp \sim \left(\frac{\tau_{0,\perp}}{\tau_Q}\right)^{\frac{1}{1+z_\perp\nu_\perp}}, \qquad \hat{\xi}_\perp \equiv \xi_\perp(\hat{\varepsilon}_\perp) \sim \xi_{0,\perp}\left(\frac{\tau_Q}{\tau_{0,\perp}}\right)^{\frac{\nu_\perp}{1+z_\perp\nu_\perp}}. \tag{5.13}$$

In relativistic-like sectors $z \simeq 1$ (propagating on g_{eff}), whereas in overdamped spectral sectors $z \simeq 2$. The anisotropy ratio

$$\mathcal{A}_{\text{KZ}} \equiv \frac{\hat{\xi}_\parallel}{\hat{\xi}_\perp} = \frac{\xi_{0,\parallel}}{\xi_{0,\perp}}\left(\frac{\tau_Q}{\tau_{0,\parallel}}\right)^{\frac{\nu_\parallel}{1+z_\parallel\nu_\parallel}}\left(\frac{\tau_{0,\perp}}{\tau_Q}\right)^{\frac{\nu_\perp}{1+z_\perp\nu_\perp}} \tag{5.14}$$

quantifies the directional bias imparted by $\hat{\chi}^{(0)}$.

Causality (emergent lightcone) bound. Defects cannot correlate beyond the *KZ horizon*

$$\ell_{\text{hor}} \equiv \int_{-\infty}^{\hat{t}} dt\; c_{\text{eff}}(t,\theta), \qquad c_{\text{eff}}(t,\theta) \in \{c_s(\theta),\, c_T,\, c_\eta(\theta)\} \le 1, \tag{5.15}$$

which imposes $\hat{\xi}_{\parallel,\perp} \lesssim \ell_{\text{hor}}$ direction-by-direction. This embeds the KZ construction within the causal cones of g^{eff}.

Defect densities and orientation statistics

In $d = 3$, the standard KZ counting adapted to anisotropy gives (up to order-one geometric factors):

$$n_{\text{wall}} \sim \hat{\xi}_\perp^{-1}, \qquad \text{(area density per volume; walls normal preferentially} \parallel \hat{\chi}^{(0)}), \tag{5.16}$$

$$\mathcal{L}_{\text{string}} \sim \hat{\xi}_\perp^{-2}, \qquad \text{(string length per volume; tangents tend to align} \parallel \hat{\chi}^{(0)}), \tag{5.17}$$

$$n_{\text{mono}} \sim \hat{\xi}_\parallel^{-1}\hat{\xi}_\perp^{-2}, \qquad \text{(monopole number per volume)}, \tag{5.18}$$

$$\epsilon_{\text{Hopf}} \sim \frac{E_{\text{Hopf}}(\hat{\xi}_\parallel, \hat{\xi}_\perp)}{\hat{\xi}_\parallel\,\hat{\xi}_\perp^2}, \qquad \text{(energy density in textures; cf. Vol. I, Ch. 4)}. \tag{5.19}$$

A minimal orientation model for strings adopts a von Mises–Fisher distribution for the unit tangent $\hat{t}$:

$$P(\hat{t}\,|\,\kappa_{\text{aniso}}) = \frac{\kappa_{\text{aniso}}}{4\pi\sinh\kappa_{\text{aniso}}}\exp\!\left[\kappa_{\text{aniso}}\,(\hat{t}\cdot\hat{\chi}^{(0)})\right], \qquad \kappa_{\text{aniso}} \propto \hat{\xi}_\perp^{-2} - \hat{\xi}_\parallel^{-2}, \tag{5.20}$$

so that $\kappa_{\text{aniso}} > 0$ biases alignment along $\hat{\chi}^{(0)}$. Analogous orientation priors apply to wall normals.

Interplay of linear seeding and KZ freezing

The instability scale $\xi_{\rm seed}$ from Eq. (5.9) and the KZ scales $\hat{\xi}_{\parallel,\perp}$ from Eqs. (5.12)–(5.13) combine to set the *initial* defect spacing:

$$\xi_{\rm init,\parallel} \sim \max\{\xi_{\rm seed},\ \hat{\xi}_{\parallel}\}, \qquad \xi_{\rm init,\perp} \sim \max\{\xi_{\rm seed},\ \hat{\xi}_{\perp}\}, \tag{5.21}$$

with subsequent coarsening controlled by curvature-driven motion and friction against the η background. The *consistency gates* inherited from Vol. I (positivity of time-kinetic blocks, subluminality, and energy positivity) ensure that the freeze-out scales remain inside the EFT domain ($k_{\rm max} \ll \Lambda_{\rm spec}$) and respect $g_{\rm eff}$ causality.

Summary and outputs for later chapters

Section 5.1 delivers: (i) the precise topological catalog for defects supported by the composite order manifold (5.1); (ii) anisotropic Kibble–Zurek scaling laws (5.12)–(5.13) tied to spectral exponents $(\nu_{\parallel,\perp}, z_{\parallel,\perp})$; (iii) directional defect densities (5.16)–(5.18) and a controlled orientation bias (5.20); and (iv) the interplay rule (5.21) between linear modulational seeding and topological freezing. These ingredients feed into Sec. 5.2 (core structure, masses, and self-interactions) and Sec. 5.3 (astrophysical constraints and observational windows).

5.2 Mass Scales, Cores, and Self–Interactions

Static energy functional in the aether frame

In the aether rest frame u^μ (Vol. I, Ch. 2), static configurations $S(\mathbf{x}) = \{\rho(\mathbf{x}), \eta(\mathbf{x}), \Phi(\mathbf{x}), \chi^I(\mathbf{x})\}$ have energy

$$E[S] \;=\; \int_{\mathbb{R}^3} d^3x\ \mathcal{H}[S, \nabla S]\,, \qquad \mathcal{H} \;=\; \mathcal{H}_2 + \mathcal{H}_4 + U_{\rm eff}(S)\,, \tag{5.22}$$

where (i) $\mathcal{H}_2$ contains quadratic gradient terms (spectral stiffness), (ii) $\mathcal{H}_4$ collects quartic "Skyrme–like" terms ensuring scale stability (Derrick), and (iii) $U_{\rm eff}$ is the effective potential of the broken phase. Anisotropy enters through the director $\hat{\chi}$:

$$\begin{aligned}\mathcal{H}_2 = \tfrac{1}{2}\Big[&Z_{\rho,\parallel}\,(\nabla_\parallel\rho)^2 + Z_{\rho,\perp}\,(\nabla_\perp\rho)^2 + Z_{\eta,\parallel}\,(\nabla_\parallel\eta)^2 + Z_{\eta,\perp}\,(\nabla_\perp\eta)^2 \\ &+ Z_{\alpha,\parallel}\,(\nabla_\parallel\alpha)^2 + Z_{\alpha,\perp}\,(\nabla_\perp\alpha)^2 + Z_\chi^\parallel\,(\nabla_\parallel\chi^I)^2 + Z_\chi^\perp\,(\nabla_\perp\chi^I)^2\Big],\end{aligned} \tag{5.23}$$

$$\mathcal{H}_4 = \tfrac{1}{2}\Big[\kappa_{\rm FS}^{(1)}\,(\partial_i n\times\partial_j n)^2 + \kappa_{\rm FS}^{(2)}\,(\partial_i\hat{\chi}\times\partial_j\hat{\chi})^2 + \cdots\Big], \qquad n = n(\Phi,\chi)\in S^2 \text{ (unit composite)}. \tag{5.24}$$

Here $\nabla_{\parallel,\perp}$ denote projections along / orthogonal to $\hat{\chi}$, and $\alpha = \arg\Phi$. The quartic sector (*Faddeev–Skyrme* type) follows from the curvature–safe EFT (Vol. I, Ch. 4, Sec. 2.6) and is crucial for stabilizing textures and knotted solitons against collapse.

Derrick scaling and virial identity. Under $\mathbf{x}\to\lambda\mathbf{x}$, the contributions scale as $E_2(\lambda) = \lambda\,E_2$, $E_4(\lambda) = \lambda^{-1}E_4$, $E_0(\lambda) = \lambda^3 E_0$. A finite–size extremum obeys the virial constraint

$$E_2 - E_4 + 3E_0 = 0 \qquad\Rightarrow\qquad \text{stable cores require a competition among } (E_2, E_4, E_0). \tag{5.25}$$

Core size estimates and tensions/masses by defect class

Let ℓ denote the characteristic core scale set by the balance of gradients, quartic stiffness and potential curvature. We list the leading scalings; detailed profiles follow from solving the reduced ODE/PDEs.

Domain walls ($\pi_0\neq 0$). For a single effective field φ (e.g. a branch of α) with $\mathcal{H}\simeq\frac{1}{2}Z_\varphi(\partial_x\varphi)^2+U(\varphi)$, the kink obeys

$$\ell_{\rm wall} \sim \sqrt{\frac{Z_\varphi}{U''(\varphi_\pm)}}\,, \qquad \sigma \;=\; \int_{-\infty}^{+\infty} dx\,\Big[Z_\varphi(\partial_x\varphi)^2\Big] \;=\; \int_{\varphi_-}^{\varphi_+} d\varphi\,\sqrt{2Z_\varphi\,U(\varphi)}\,. \tag{5.26}$$

Anisotropy enters via $Z_\varphi \to Z_{\varphi,\perp}$ when the normal is transverse to $\hat{\chi}$, etc., thus orienting minimal–tension walls.

Straight vortices / strings ($\pi_1 \neq 0$). With cylindrical ansatz $\alpha = n\phi$, the phase energy density contains

$$\mathcal{H}_\alpha \supset \tfrac{1}{2} Z_{\alpha,\perp} \frac{n^2}{r^2} - J^\phi_{\text{comp}}\,, \qquad J^\phi_{\text{comp}} \equiv \text{compensating current from the response field } \chi^\mu, \tag{5.27}$$

so that the long–range $1/r^2$ is screened by an emergent spectral gauge mass m_{spec} (Vol. I, Ch. 4). The tension per unit length saturates:

$$\mathcal{T}_n \;\simeq\; \pi Z_{\alpha,\perp}\, n^2 \,\ln\!\Big(1 + m^2_{\text{spec}} R^2\Big) \;\longrightarrow\; \mu_n \quad (R \to \infty)\,, \tag{5.28}$$

with core size $\xi_{\text{vtx}} \sim \max\{m^{-1}_{\text{spec}},\ \sqrt{Z_{\alpha,\perp}/U''_{\text{eff}}}\}$. Anisotropic $Z_{\alpha,\parallel} \neq Z_{\alpha,\perp}$ deforms cores elliptically.

Monopoles / hedgehogs ($\pi_2 \neq 0$). A spherically–reduced composite $n^a(\mathbf{x})$ with $\mathcal{H} \supset \frac{1}{2}\kappa_2(\partial_i n)^2 + \frac{1}{2}\kappa_4(\partial_i n \times \partial_j n)^2 + U(n)$ gives (by minimizing $E(\ell) \sim 4\pi[\kappa_2 \ell + \kappa_4/\ell] + \frac{4}{3}\pi \ell^3 \bar{U}$):

$$\ell_{\text{mono}} \;\sim\; \sqrt{\frac{\kappa_4}{\kappa_2}}\,\left[1 + \mathcal{O}\!\Big(\frac{\bar{U}\,\ell^2_{\text{mono}}}{\kappa_2}\Big)\right], \qquad M_{\text{mono}} \;\simeq\; 8\pi\sqrt{\kappa_2\kappa_4} \;+\; \mathcal{O}(\bar{U}\,\ell^3_{\text{mono}}). \tag{5.29}$$

Hopfions / Skyrmion–like textures ($\pi_3 \neq 0$). In the Faddeev–Skyrme class, the energy scales with the Hopf charge Q_H as

$$E_{\text{Hopf}} \;\simeq\; C_{\text{H}}\, \kappa_2^{1/2}\, \kappa_4^{1/2}\, |Q_H|^{3/4}\,, \qquad \ell_{\text{Hopf}} \;\sim\; \left(\frac{\kappa_4}{\kappa_2}\right)^{1/2} |Q_H|^{1/4}, \tag{5.30}$$

with $C_{\text{H}} = \mathcal{O}(10)$ depending weakly on knot type (Vol. I, Ch. 4). Anisotropy modifies the prefactor via directional stiffnesses in (5.23).

Effective two–body self–interactions

Long–range (Yukawa–like) spectral exchange. Exchange of a light spectral mediator of mass m_{spec} and coupling g_{spec} produces

$$V_{\text{LR}}(\mathbf{R}; \Omega_1, \Omega_2) \;\simeq\; \frac{g^2_{\text{spec}}}{4\pi}\, \frac{e^{-m_{\text{spec}} R}}{R}\; \mathcal{S}(\Omega_1, \Omega_2, \hat{\mathbf{R}}\cdot\hat{\chi})\,, \tag{5.31}$$

where $\Omega_{1,2}$ denote soliton orientations and internal phases, and $\mathcal{S}$ encodes the angular anisotropy inherited from $\hat{\chi}$. The emergent gravitating tail adds a universal $-G_{\text{eff}} M_1 M_2/R$ at very large R.

Short–range (core) interaction. Core overlap produces a repulsive contact characterized by a scattering length $a_0 \sim \alpha_0\, \ell$ ($\alpha_0 = \mathcal{O}(1)$), giving the s–wave Born estimate

$$\sigma_{\text{SR}} \;\simeq\; 4\pi a_0^2 \;\sim\; 4\pi\, \alpha_0^2\, \ell^2. \tag{5.32}$$

For anisotropic cores, one may refine to an orientation–averaged $\bar{a}_0$.

Total transport cross section and σ/M. To leading order (weak coupling, low v), the transport cross section per unit mass reads

$$\frac{\sigma_T}{M_{\text{sol}}} \;\simeq\; \frac{4\pi\, \alpha_0^2\, \ell^2}{M_{\text{sol}}} \;+\; \frac{g^4_{\text{spec}}}{4\pi\, \mu^2\, v^4}\, \mathcal{F}\!\left(\frac{m_{\text{spec}}}{\mu v},\ \langle \mathcal{S} \rangle\right), \tag{5.33}$$

where μ is the reduced mass, v the relative velocity, and $\mathcal{F}$ a known Yukawa form factor (with anisotropy averaged into $\langle\mathcal{S}\rangle$). Using the mass scalings above,

$$\frac{\sigma_T}{M_{\rm sol}} \sim \begin{cases} \dfrac{4\pi\alpha_0^2\,\ell^2}{\rho_0\,\ell^3} \;\propto\; \ell^{-1}, & \text{lumps (non-topological)} \\ \dfrac{4\pi\alpha_0^2\,\xi_{\rm vtx}^2}{\mu_n L} \;\propto\; \dfrac{\xi_{\rm vtx}^2}{\mu_n}, & \text{strings (per length } L) \\ \dfrac{4\pi\alpha_0^2\,\ell_{\rm mono}^2}{8\pi\sqrt{\kappa_2\kappa_4}} \;\propto\; \dfrac{\sqrt{\kappa_4/\kappa_2}}{\sqrt{\kappa_2\kappa_4}}, & \text{monopoles} \\ \dfrac{4\pi\alpha_0^2\,\ell_{\rm Hopf}^2}{C_{\rm H}\kappa_2^{1/2}\kappa_4^{1/2}|Q_H|^{3/4}} \;\propto\; |Q_H|^{-1/4}. \end{cases}$$

These relations will be mapped to astrophysical constraints in Sec. 5.3.

Orientation dependence and multipoles

Because the moduli space includes orientation $\mathcal{R} \in SO(3)$ and internal phases, the leading two–body potential admits a tensor decomposition

$$V(\mathbf{R},\mathcal{R}_1,\mathcal{R}_2) \;=\; V_0(R) \;+\; V_2(R)\left[(\hat{\mathbf{R}}\cdot\hat{\chi})^2 - \tfrac{1}{3}\right] \;+\; W(R)\,\mathrm{Tr}\big(\mathcal{R}_1^T\mathcal{R}_2\big) \;+\; \cdots\,, \tag{5.34}$$

leading to alignment torques and anisotropic scattering. The coefficients (V_0, V_2, W) are calculable from the worldvolume EFT matched to form factors (Vol. I, Sec. 6.14), and inherit $m_{\rm spec}$–dependent Yukawa falloffs.

Consistency gates (static sector)

All estimates above are contingent on the gates:

- **(S1) Positivity:** time–kinetic Hessians $\succ 0$ (no ghosts); energy density $\mathcal{H} \geq 0$.
- **(S2) Hyperbolicity:** principal symbols respect $g_{\rm eff}$ cones; $0 < c_{\perp,\parallel}^2 \leq 1$.
- **(S3) Derrick/virial:** quartic sector $\mathcal{H}_4$ present when required by (5.25).
- **(S4) EFT validity:** cores satisfy $\ell^{-1} \ll \Lambda_{\rm spec}$; long–range exchange within EFT cutoffs.
- **(S5) Anisotropy control:** $|Z_{\bullet,\parallel} - Z_{\bullet,\perp}|/Z_\bullet \ll 1$ when confronted with isotropy-sensitive data.

Summary. We provided a unified static energy functional (5.22) with anisotropic stiffnesses, derived core sizes and mass/tension scalings for walls (5.26), strings (5.28), monopoles (5.29), and Hopfions (5.30), and obtained analytic estimates for self–interaction cross sections (5.32)–(5.33). These will be confronted with astrophysical windows in Sec. 5.3.

5.3 Astrophysical Bounds and Observational Windows

Relic abundance from formation physics

Let t_* denote the end of the instability/transition that seeds defects (Sec. 5.1). With a freeze–out correlation length ξ_* and a formation fraction $f_* \leq 1$ per correlation domain, the initial comoving number density of objects of type X (wall patches, strings per cell, monopoles, Hopfions, or non–topological lumps) is

$$n_X^{\rm com}(t_*) \;\simeq\; f_*\,\xi_*^{-d_X}\,, \qquad d_X = \begin{cases} 3 & \text{lumps / monopoles / Hopfions,} \\ 2 & \text{strings (per unit length),} \\ 1 & \text{walls (per unit area).} \end{cases} \tag{5.35}$$

Redshifting to today,

$$n_X(t_0) \;=\; n_X^{\rm com}(t_*)\,a_*^{-3}\,, \qquad \rho_X(t_0) \;=\; n_X(t_0)\,M_X\,, \qquad \Omega_X \equiv \frac{\rho_X(t_0)}{\rho_{\rm crit}}\,. \tag{5.36}$$

Using the mass/tension scalings of Sec. 5.2 (e.g. $M_{\rm mono} \simeq 8\pi\sqrt{\kappa_2\kappa_4}$, $E_{\rm Hopf} \propto |Q_H|^{3/4}$, $\mathcal{T}_n$ for strings), the requirement $\Omega_X \simeq \Omega_{\rm DM} \approx 0.26$ constrains (ξ_*, f_*) and the stiffness parameters $(Z_\bullet, \kappa_{2,4}, U)$. In practice we scan the *formation window*

$$(\xi_*, f_*) \;\; \text{from Kibble–Zurek} \;\Rightarrow\; \Omega_X(\text{micro}) \overset{!}{\simeq} 0.26\,, \tag{5.37}$$

discarding wall networks that survive beyond recombination (see gates below).

Self–interaction cross section across velocity scales

From the worldvolume/EFT matching (Sec. 5.2), the transport cross section per unit mass reads

$$\frac{\sigma_T}{M}(v) \;\simeq\; \underbrace{\frac{4\pi\,\alpha_0^2\,\ell^2}{M}}_{\text{core contact}} \;+\; \underbrace{\frac{g_{\rm spec}^4}{4\pi\,\mu^2\,v^4}\,\mathcal{F}\!\left(\frac{m_{\rm spec}}{\mu v},\;\langle\mathcal{S}\rangle\right)}_{\text{Yukawa–like exchange, velocity dependent}}\,, \tag{5.38}$$

with reduced mass μ, core scale ℓ and anisotropy factor $\langle\mathcal{S}\rangle$ (orientation–averaged). Astrophysical inferences favor a *velocity–dependent* σ_T/M, roughly:

$$\text{Dwarf scales: } v \sim 10{-}50\ \mathrm{km\,s^{-1}}, \qquad \frac{\sigma_T}{M}\Big|_{\rm dwarf} \sim \mathcal{O}(0.5{-}10)\ \mathrm{cm^2\,g^{-1}}, \tag{5.39}$$

$$\text{Clusters: } v \sim 1000\ \mathrm{km\,s^{-1}}, \qquad \frac{\sigma_T}{M}\Big|_{\rm cluster} \lesssim \mathcal{O}(0.1{-}1)\ \mathrm{cm^2\,g^{-1}}. \tag{5.40}$$

The Yukawa term in (5.38) falls with v, naturally accommodating (5.39)–(5.40) for mediator ranges $m_{\rm spec}^{-1} \sim (1{-}100)\ \mathrm{kpc}/(\mu v)$ and couplings $g_{\rm spec}$ in the perturbative window. Core contact sets a floor $\propto \ell^2/M$; using $M \sim \rho_0\ell^3 + \cdots$ for non–topological lumps gives $\sigma_T/M \propto \ell^{-1}$, favoring moderately *large* cores for dwarfs.

Relaxation time and core formation. The collisional relaxation time at density ρ and velocity v,

$$t_{\rm rel} \;\sim\; \frac{1}{\rho\,(\sigma_T/M)\,v}\,, \tag{5.41}$$

must satisfy $t_{\rm rel} \lesssim t_H$ (Hubble time) in dwarfs to form cores, yet $t_{\rm rel} \gtrsim t_H$ in clusters to preserve ellipticity—again suggestive of a velocity–dependent sector with $m_{\rm spec}$ tuned to galactic scales.

Lensing, streams, and small–scale structure

Strong lensing substructure. Dark solitons (lumps/Hopfions) act as subhalos with concentration set by their core physics; their *projected* surface densities and finite–size lenses can induce flux ratio anomalies and astrometric shifts. Forward models use the soliton density profile $\rho_{\rm sol}(r;\ell,\kappa_{2,4})$ and abundance n_X from (5.35) to predict the substructure power spectrum $P_\kappa(k_\perp)$.

Stellar streams and disk heating. Perturbations by passing solitons leave gaps in cold stellar streams; the rate scales with $n_X\,v\,\pi b_{\rm max}^2$, where $b_{\rm max}$ is set by the impulse approximation with the soliton's finite core. Stream catalogs bound $n_X M^2$ at $\mathcal{O}(10^6{-}10^8)M_\odot$ mass scales—thus constraining (ξ_*, f_*) independently of σ_T/M.

Lyman–α and subhalo counts. If the mediator causes sizable damping of small–scale power (via early–time drag among defects/solitons), the cut–off $k_{\rm damp}$ maps to $(g_{\rm spec}, m_{\rm spec})$; data favor $k_{\rm damp} \gtrsim \mathcal{O}(10\,h\,\mathrm{Mpc^{-1}})$ unless compensated by later growth.

Extended defects: strings and walls

String networks. A residual network of spectral strings behaves as conical defects in $g_{\rm eff}$ with deficit angle

$$\Delta\theta \;\simeq\; 8\pi\,G_{\rm eff}\,\mu\,, \qquad \mu \equiv \mathcal{T}_n, \tag{5.42}$$

producing line–like lensing and a stochastic background from intercommutation. Bounds on μ (PTA/CMB/lensing) translate to $\mu/M_{\rm eff}^2 \ll 1$. Our *string gate* requires either (i) inflationary dilution of long strings or (ii) sufficiently small μ and reconnection probability to evade current limits, while allowing small closed loops as a subdominant DM component if stable.

Domain walls. Scaling wall networks overclose or imprint large CMB anisotropies unless biased. The *wall gate* demands: (a) no long–lived scaling wall network after recombination; (b) either bias $\Delta U \neq 0$ between vacua (driving collapse) or inflationary dilution. Localized wall segments (finite membranes) may survive as rare objects provided σ and number density satisfy $\Omega_{\rm wall} \ll 0.01$.

Tensor sector and multi–messenger bounds

Tensor propagation in $g_{\rm eff}$ obeys $c_T^2 = 1 + \delta c_T(\mathcal{S})$ (Vol. II, Sec. 4.2). Multi–messenger timing bounds enforce $|\delta c_T| \ll 1$, constraining curvature couplings in the Palatini sector and any large–scale alignment from $\hat{\chi}$. Additionally, the stochastic GW background from strings/oscillons must stay below observational limits—yielding upper bounds on (μ, $n_{\rm loop}$) and on η–sector oscillations after exit.

Mapping observational windows to spectral parameters

Observable / Bound	Window / Inequality	Spectral–parameter map
Relic abundance	$\Omega_X \simeq 0.26$	(ξ_*, f_*) & $(\kappa_{2,4}, Z_\bullet, U)$ via M_X
Dwarfs cores	$t_{\rm rel} \lesssim t_H$	$\sigma_T/M(v{\sim}30\ {\rm km/s}) \gtrsim 1\ {\rm cm}^2/{\rm g}$
Clusters shapes	$t_{\rm rel} \gtrsim t_H$	$\sigma_T/M(v{\sim}1000\ {\rm km/s}) \lesssim 0.1$–$1$
Strings (lens/GW)	$\Delta\theta \ll$ arcsec; $\Omega_{\rm GW}$ small	$\mu/M_{\rm eff}^2 \ll 1$; reconnection efficient
Walls (CMB)	No scaling network	Bias $\Delta U \neq 0$ or inflationary dilution
GW speed	$\lvert c_T - 1\rvert \ll 1$	Palatini couplings $\Rightarrow \delta c_T(\mathcal{S}) \approx 0$
Lyman–α	$k_{\rm damp} \gtrsim 10\, h\,{\rm Mpc}^{-1}$	$(g_{\rm spec}, m_{\rm spec})$ small drag early

Consistency gates (astrophysical sector)

- **(A1) Abundance:** $\Omega_{\rm wall} + \Omega_{\rm string} + \Omega_{\rm lump/Hopf} \simeq 0.26$ with each subcomponent below dedicated bounds.
- **(A2) Velocity dependence:** $\sigma_T/M(v)$ satisfies (5.39)–(5.40) via (5.38) without violating perturbativity or EFT validity.
- **(A3) Non–disruption:** Substructure predictions (lensing/streams) consistent with survey statistics for chosen (ξ_*, f_*, ℓ).
- **(A4) Tensor safety:** $|\delta c_T| \ll 1$; stochastic GW from networks/oscillations below current bounds.
- **(A5) Restoration:** Large–scale isotropy restored post–inflation; any residual director alignment yields g_* well within limits.

Summary. The dark–soliton hypothesis emerging from the spectral aether admits a tightly constrained but ample parameter space. Relic abundance fixes (ξ_*, f_*) given the microphysics of Sec. 5.2; velocity–dependent self–interactions from a finite–range mediator reconcile dwarf cores with cluster shapes; extended defects must be inflated away or highly suppressed. This section furnishes the translation layer from observational windows to the spectral couplings $(g_{\rm spec}, m_{\rm spec}, \kappa_{2,4}, Z_\bullet, U, M_{\rm eff})$, enabling quantitative confrontation with data in later chapters.

Part III

Exit, Reheating, and the Early Thermal History

Chapter 6

Exit, Reheating, and Early Thermal History

6.1 Dissipation Rates Γ_η and Energy Transfer

This section derives the dissipation rate Γ_η of the spectral–viscosity condensate and quantifies energy transfer from the rolling/oscillating background $\bar{\eta}(t)$ into (i) matter excitations of the Φ sector and (ii) structural modes of the aether $S = \{\rho, \chi^\mu, T, \ldots\}$. We work in the comoving–aether frame u^μ and with the emergent metric $g^{\rm eff}_{\mu\nu}$ of Volume I, so that dots denote $u^\mu \nabla_\mu$.

6.1.1 Effective Lagrangian, friction, and the continuity system

We start from the effective sector (cf. Vol. I, Ch. 3; Sec. **??**)

$$\mathcal{L}_\eta \;=\; \frac{\kappa_\eta(\mathcal{S})}{2}\, g^{\mu\nu}_{\rm eff}\, \nabla_\mu\eta\, \nabla_\nu\eta - V_\eta(\eta) - \frac{\Gamma_\eta(\mathcal{S})}{2}\, (u^\mu\nabla_\mu\eta)^2, \tag{6.1}$$

where $\kappa_\eta > 0$ sets the canonical normalization and $\Gamma_\eta \geq 0$ encodes coarse–grained friction emerging from microscopic couplings between η and other sectors. Varying (6.1) in the homogeneous limit $\eta = \bar{\eta}(t)$ yields

$$\kappa_\eta\, \ddot{\bar{\eta}} + (3H\,\kappa_\eta + \dot{\kappa}_\eta + \Gamma_\eta)\, \dot{\bar{\eta}} + V'_\eta(\bar{\eta}) = 0, \tag{6.2}$$

with $H = \frac{1}{3}\nabla_\mu u^\mu$. The corresponding energy densities and pressures are

$$\rho_\eta = \tfrac{\kappa_\eta}{2}\dot{\bar{\eta}}^{\,2} + V_\eta, \qquad p_\eta = \tfrac{\kappa_\eta}{2}\dot{\bar{\eta}}^{\,2} - V_\eta. \tag{6.3}$$

Using (6.2) one obtains the fluid–level continuity equations

$$\dot{\rho}_\eta + 3H(\rho_\eta + p_\eta) \;=\; -\Gamma_\eta\, \dot{\bar{\eta}}^{\,2}, \qquad \dot{\rho}_X + 3H(\rho_X + p_X) \;=\; +\Gamma_\eta\, \dot{\bar{\eta}}^{\,2}, \tag{6.4}$$

where X collectively denotes the recipient sectors (Φ quanta and structural excitations). Equation (6.4) makes manifest the gate of nonnegative entropy production: $\Gamma_\eta \geq 0 \Rightarrow \sigma_{\rm diss} \propto \Gamma_\eta \dot{\bar{\eta}}^{\,2} \geq 0$ (cf. Vol. I, Sec. 3.9).

6.1.2 Microscopic origin of Γ_η: influence functionals and Kubo formula

Let $\mathcal{L} = \mathcal{L}_\eta + \mathcal{L}_\Phi + \mathcal{L}_S + \mathcal{L}_{\rm int}$ with representative, symmetry–allowed interactions (to leading order)

$$\mathcal{L}_{\rm int} \supset -\frac{1}{2} g_{\eta\Phi}(\mathcal{S})\, \eta\, |\Phi|^2 - h_{\eta S}(\mathcal{S})\, \eta\, \mathcal{O}_S - \frac{1}{4}\lambda_{\eta\Phi}\, \eta^2 |\Phi|^2 + \ldots, \tag{6.5}$$

where $\mathcal{O}_S$ is a scalar composite built from structural fields (e.g. gradients of ρ or χ consistent with the internal symmetries). Integrating out Φ and S at quadratic order in the couplings gives a nonlocal influence action for η,

$$S_{\rm inf}[\eta] \;=\; \frac{1}{2}\!\int\! d^4x\, d^4x'\; \eta(x)\, \Sigma_R(x,x')\, \eta(x') \;+\; \frac{i}{2}\!\int\! d^4x\, d^4x'\; \eta(x)\, \mathcal{N}(x,x')\, \eta(x') + \cdots, \tag{6.6}$$

with Σ_R the retarded self–energy and $\mathcal{N}$ the noise kernel. In Fourier space (comoving frame), the low–frequency limit of the retarded kernel defines the friction:

$$\Gamma_\eta \;=\; -\lim_{\omega\to 0}\frac{1}{\omega}\,\mathrm{Im}\;\Sigma_R(\omega,\mathbf{k}{=}0)\,. \tag{6.7}$$

Fluctuation–dissipation implies $\mathcal{N}(\omega) = 2\coth(\omega/2T_{\rm eff})\,\omega\,\Gamma_\eta(\omega)$ for an effective bath temperature $T_{\rm eff}$ (vacuum limit $T_{\rm eff}\to 0$). Equation (6.7) ensures $\Gamma_\eta \geq 0$ if the spectral density of the bath operators is positive.

Perturbative estimate (quasi–particle picture). Near the minimum of V_η, small oscillations of η have mass $m^2_{\eta,\rm eff} = V''_\eta(\bar\eta_\star)$. Treating η as a quasi–particle, the partial widths at rest read, schematically,

$$\Gamma(\eta\to\Phi\Phi) = \frac{g^2_{\eta\Phi,\rm eff}}{8\pi\, m_{\eta,\rm eff}}\sqrt{1-\frac{4m^2_\Phi}{m^2_{\eta,\rm eff}}}\;\Theta(m_{\eta,\rm eff}-2m_\Phi), \tag{6.8}$$

$$\Gamma(\eta\to SS) = \frac{h^2_{\eta S,\rm eff}}{8\pi\, m_{\eta,\rm eff}}\sqrt{1-\frac{4m^2_S}{m^2_{\eta,\rm eff}}}\;\Theta(m_{\eta,\rm eff}-2m_S), \tag{6.9}$$

with effective couplings $g_{\eta\Phi,\rm eff}, h_{\eta S,\rm eff}$ after canonical normalization by κ_η and the field–space metrics of Vol. I. The total microscopic rate $\Gamma^{\rm pert}_\eta = \sum_i \Gamma_i$ matches the Kubo definition (6.7) in the narrow–width limit.

6.1.3 Nonperturbative transfer: parametric amplification and Floquet theory

When $\bar\eta(t)$ passes through a regime of coherent oscillations (post–slow-roll), the Φ modes acquire a time–dependent mass

$$m^2_\Phi(t) = m^2_{\Phi,0} + g_{\eta\Phi,\rm eff}\,\bar\eta(t), \qquad \bar\eta(t)\simeq \eta_A\,\cos(m_{\eta,\rm eff}t)\,, \tag{6.10}$$

and satisfy (in Fourier space, neglecting Hubble friction for sub–Hubble modes)

$$\delta\ddot\Phi_{\mathbf{k}} + \left[\omega_k^2 + 2q\,\cos(2z)\right]\delta\Phi_{\mathbf{k}} = 0, \quad \omega_k^2 = \frac{k^2}{a^2 m^2_{\eta,\rm eff}} + \frac{m^2_{\Phi,0}}{m^2_{\eta,\rm eff}}, \quad q = \frac{g_{\eta\Phi,\rm eff}\,\eta_A}{2m^2_{\eta,\rm eff}}, \quad z = m_{\eta,\rm eff}t. \tag{6.11}$$

Equation (6.11) is a Mathieu–type equation exhibiting resonance bands with Floquet exponent $\mu_k > 0$; the occupation number grows as $n_k \propto e^{2\mu_k z}$. Broad resonance ($q \gtrsim 1$) yields explosive, nonperturbative energy transfer ("preheating"). Backreaction shuts the resonance when the induced $\langle|\Phi|^2\rangle$ back–reacts on $m^2_\Phi(t)$ and when rescattering redistributes energy among modes; both are included by Hartree–type replacements $m^2_\Phi \to m^2_{\Phi,0} + g_{\eta\Phi,\rm eff}(\bar\eta + \langle\delta\eta\rangle) + \lambda_\Phi\langle|\Phi|^2\rangle$ and by adding collision integrals in the mode–kinetic description.

An analogous analysis applies to structural channels S when $h_{\eta S,\rm eff}\bar\eta(t)$ modulates the quadratic kernel of δS, subject to the hyperbolicity/causality gates of Vol. I.

6.1.4 Boltzmann system and sourced production

At coarse–grained level, the energy flow can be tracked with

$$\dot\rho_\eta + 3H(\rho_\eta + p_\eta) = -\,\Gamma^{\rm eff}_\eta\,\dot{\bar\eta}^{\,2} - \mathcal{Q}_{\rm prh}, \tag{6.12}$$

$$\dot\rho_\Phi + 3H(\rho_\Phi + p_\Phi) = +\,\mathcal{C}_{\eta\to\Phi} + \mathcal{Q}^{(\Phi)}_{\rm prh} - \mathcal{C}_{\Phi\leftrightarrow\rm bath}, \tag{6.13}$$

$$\dot\rho_S + 3H(\rho_S + p_S) = +\,\mathcal{C}_{\eta\to S} + \mathcal{Q}^{(S)}_{\rm prh} - \mathcal{C}_{S\leftrightarrow\rm bath}, \tag{6.14}$$

where $\Gamma^{\rm eff}_\eta$ captures perturbative friction, $\mathcal{Q}_{\rm prh}$ accounts for resonant (preheating) transfer, and the $\mathcal{C}$'s are collision integrals that thermalize each sector. In the perturbative regime, $\mathcal{Q}_{\rm prh}\to 0$ and $\mathcal{C}_{\eta\to X} = \Gamma_{\eta\to X}\dot{\bar\eta}^{\,2}$. In the resonant regime, $\mathcal{Q}^{(X)}_{\rm prh} \sim \int \frac{d^3k}{(2\pi)^3}\,2\mu_k\,\omega_k\,n_k$ with μ_k from Floquet analysis.

6.1.5 Matching prescriptions and parameter hierarchy

For practical scans we adopt:

1. **Perturbative (slow–roll/weak–mixing):** $\Gamma_\eta^{\rm eff} = \sum_i \Gamma_i(\bar\eta)$ with Γ_i computed from tree–level widths using canonical fields ($\kappa_\eta \to 1$ normalization) and evaluated at $m_{\eta,\rm eff}(\bar\eta)$.
2. **Resonant:** solve (6.11) for the leading channels, compute $\mu_k(\bar\eta)$ and integrate mode energy growth until backreaction shuts resonance; translate to an effective $\mathcal{Q}_{\rm prh}$ and an average $\Gamma_\eta^{\rm eff}(t)$ for use in (6.12).
3. **Consistency gates:** (i) positivity/causality: $\Gamma_\eta \geq 0$, principal symbols hyperbolic; (ii) gauge/phase symmetries preserved in (6.5); (iii) restoration: anisotropies from χ^μ redshift/thermalize so that late–time $c_T \to 1$, $c_s(\theta) \to c_s^{\rm iso}$.

6.1.6 Energy partition and diagnostic observables

Define branching ratios $B_\Phi \equiv (\dot\rho_\Phi)_{\rm inj}/(\Gamma_\eta^{\rm eff}\dot{\bar\eta}^2 + \mathcal{Q}_{\rm prh})$ and $B_S = 1 - B_\Phi$. The relative partition controls:

- the reheating temperature $T_{\rm rh}$ and effective relativistic DOF g_* (next section),
- the level of residual isocurvature between matter and structural backgrounds,
- possible production of relic defects or Φ–particles (dark components) during reheating.

A useful integral diagnostic is

$$\Xi_X \;\equiv\; \int_{t_{\rm exit}}^{t_{\rm rh}} \frac{(\dot\rho_X)_{\rm inj}}{3H(\rho_\eta + p_\eta)}\, dt, \qquad X \in \{\Phi, S\}, \tag{6.15}$$

which measures the cumulative injection relative to Hubble–diluted η enthalpy and helps classify exit scenarios (smooth vs. abrupt).

Summary. The dissipation rate Γ_η is fixed by the low–frequency limit of the retarded self–energy of η (Kubo relation), admits a particle–physics interpretation through partial widths when $V_\eta'' > 0$, and can be vastly enhanced by parametric resonance. The coarse–grained energy transfer equations (6.4)–(6.14) provide the bridge to reheating phenomenology and set the initial conditions for the early thermal history, to which we now turn.

6.2 Reheating Temperature, Relics, and BBN Compatibility

We now translate the dissipation dynamics of Sec. 6.1 into thermodynamic initial conditions for the hot epoch, derive the reheating temperature $T_{\rm rh}$, and state the Big–Bang Nucleosynthesis (BBN) and extra–radiation ($\Delta N_{\rm eff}$) gates that constrain the parameter space $(V_\eta, \kappa_\eta, M_{\rm eff}, \Gamma_\eta, g_{\eta\Phi}, h_{\eta S})$.

6.2.1 Definition of $T_{\rm rh}$ and end of reheating

Radiation domination sets in when the radiation energy density ρ_R overtakes the η–condensate energy. In the instantaneous approximation, the end of reheating occurs at

$$H(t_{\rm rh}) \;\simeq\; \Gamma_\eta^{\rm eff}(t_{\rm rh}), \qquad \rho_R(T_{\rm rh}) \;=\; \frac{\pi^2}{30}\, g_*(T_{\rm rh})\, T_{\rm rh}^4, \qquad H^2 \;=\; \frac{\rho_R}{3\, M_{\rm eff}^2(\rho)}\,, \tag{6.16}$$

which gives

$$T_{\rm rh} \;\simeq\; \xi_{\rm rh} \left(\frac{90}{\pi^2 g_*(T_{\rm rh})}\right)^{1/4} \sqrt{M_{\rm eff}(\rho_{\rm rh})\, \Gamma_\eta^{\rm eff}(t_{\rm rh})}\,. \tag{6.17}$$

The factor $\xi_{\rm rh} \in (0.5, 1)$ encodes the departure from instantaneous reheating due to a nontrivial equation of state during the decay epoch (see below). Entropy density is

$$s(T) \;=\; \frac{2\pi^2}{45}\, g_{*s}(T)\, T^3, \qquad \text{and} \qquad a^3 s \text{ is conserved for } T < T_{\rm rh} \text{ (in the visible sector).} \tag{6.18}$$

6.2.2 Equation of state during reheating and e–folds $N_{\rm reh}$

Let $\bar{w}_{\rm reh}$ denote the average equation of state while the η energy redshifts and decays. Energy density scales as

$$\rho(a) \;=\; \rho_{\rm end}\left(\frac{a}{a_{\rm end}}\right)^{-3(1+\bar{w}_{\rm reh})}, \qquad N_{\rm reh} \equiv \ln\frac{a_{\rm rh}}{a_{\rm end}} \;=\; \frac{1}{3(1+\bar{w}_{\rm reh})}\,\ln\frac{\rho_{\rm end}}{\rho_{\rm rh}}. \tag{6.19}$$

Using $\rho_{\rm rh} = (\pi^2/30) g_* T_{\rm rh}^4$, one can write

$$N_{\rm reh} \;=\; \frac{1}{3(1+\bar{w}_{\rm reh})}\,\ln\!\left[\frac{30\,\rho_{\rm end}}{\pi^2 g_*(T_{\rm rh})\,T_{\rm rh}^4}\right]. \tag{6.20}$$

The maximal temperature $T_{\rm max}$ reached during the first oscillations can exceed $T_{\rm rh}$; parametrically

$$T_{\rm max} \;=\; \xi_{\rm max}\,(M_{\rm eff}\,H_{\rm end})^{1/4}\,T_{\rm rh}^{1/2}, \qquad \xi_{\rm max} = \mathcal{O}(1), \tag{6.21}$$

which matters for production of heavy relics even if they cannot be thermally populated at $T_{\rm rh}$.

6.2.3 BBN gate and variation of $M_{\rm eff}$

Successful nucleosynthesis imposes two robust constraints:

$$T_{\rm rh} \gtrsim 4\text{–}5\,\text{MeV} \quad \text{(thermalization before BBN)}, \qquad \left|\frac{G_{\rm eff}(T_{\rm BBN}) - G_{\rm N}}{G_{\rm N}}\right| \;=\; \left|\frac{M_{\rm eff}^{-2}(T_{\rm BBN}) - M_{\rm Pl}^{-2}}{M_{\rm Pl}^{-2}}\right| \lesssim \mathcal{O}(10^{-1})\,. \tag{6.22}$$

The second inequality encodes the bound on the effective Planck mass evolution between BBN and today; in the present framework it constrains the ρ–dependence $M_{\rm eff}(\rho)$ to be quasi-constant by the BBN epoch.

6.2.4 Extra radiation and $\Delta N_{\rm eff}$ from structural channels

If a fraction of the released energy is dumped into an *invisible* (decoupled) structural sector S with temperature T_S after reheating, its contribution to the radiation density at BBN/CMB is

$$\Delta N_{\rm eff} \;=\; \frac{8}{7}\left(\frac{11}{4}\right)^{4/3}\frac{\rho_S}{\rho_\gamma}\bigg|_{\rm BBN/CMB}, \qquad \frac{\rho_S}{\rho_\gamma} \;=\; \left(\frac{T_S}{T_\gamma}\right)^4\frac{g_*(S)}{2}, \tag{6.23}$$

where $g_*(S)$ counts the relativistic DOF in S. If S decouples immediately at reheating with energy fraction $f_S \equiv \rho_S/(\rho_S+\rho_{\rm vis})|_{\rm rh}$, then

$$\left(\frac{T_S}{T_\gamma}\right)_{\rm BBN} \;=\; f_S^{1/4}\left(\frac{g_{*s}^{\rm vis}(\text{BBN})}{g_{*s}^{\rm vis}(T_{\rm rh})}\right)^{1/3}, \qquad \Rightarrow \quad \Delta N_{\rm eff} \;=\; \frac{8}{7}\left(\frac{11}{4}\right)^{4/3}\frac{g_*(S)}{2}\,f_S\left(\frac{g_{*s}^{\rm vis}(\text{BBN})}{g_{*s}^{\rm vis}(T_{\rm rh})}\right)^{4/3}, \tag{6.24}$$

to leading order (assuming $g_*(S)$ constant thereafter). Current bounds $\Delta N_{\rm eff} \lesssim \mathcal{O}(0.3)$ imply an upper limit on the invisible branching f_S for given $g_*(S)$ and $T_{\rm rh}$.

6.2.5 Thermal and nonthermal relics from Φ and structural sectors

Thermal freeze-out (visible/mixed sector). For a stable Φ species with annihilation cross section $\langle\sigma v\rangle_{\rm eff}$ in the visible bath,

$$\Omega_\Phi h^2 \;\simeq\; 0.12\left(\frac{3\times10^{-26}\,\text{cm}^3/\text{s}}{\langle\sigma v\rangle_{\rm eff}}\right)\times\mathcal{R}(M_{\rm eff}), \tag{6.25}$$

where $\mathcal{R}(M_{\rm eff})$ parametrizes the (typically mild) replacement $M_{\rm Pl}\to M_{\rm eff}$ in the freeze-out condition. If Φ thermalizes only partially (due to small $g_{\eta\Phi}$), the relic density is reduced accordingly.

Freeze-in (feeble couplings). For Φ never reaching equilibrium, the yield obeys

$$Y_{\rm FI} \equiv \frac{n_\Phi}{s} = \int_{T_0}^{T_{\rm rh}} \frac{\gamma_{\rm prod}(T)}{s(T)\,H(T)\,T}\,dT, \qquad \gamma_{\rm prod} \sim n_a n_b \langle\sigma v\rangle_{ab\to\Phi\Phi} \propto g_{\rm eff}^2\,T^n, \tag{6.26}$$

with n set by the operator dimension; $Y_{\rm FI} \propto T_{\rm rh}^{n-1}$ so that high $T_{\rm rh}$ enhances freeze-in. Production at $T_{\rm max}$ (Sec. 6.2) can dominate if n is large.

Nonthermal solitonic relics (structural sector). If dark solitons (Sec. 5) are produced during or after reheating, their present abundance is

$$\Omega_{\rm sol}h^2 = \frac{M_{\rm sol}\,n_{\rm sol}(T_0)}{\rho_c/h^2} = \frac{M_{\rm sol}}{\rho_c/h^2}\left(\frac{a_{\rm rh}}{a_0}\right)^3 n_{\rm sol}(T_{\rm rh}), \tag{6.27}$$

with $n_{\rm sol}(T_{\rm rh})$ set by the freeze-out correlation length ξ and subsequent annihilation/self–interaction history. Constraints on $\sigma_{\rm self}/M_{\rm sol}$ from clusters and halo shapes must be rechecked for the post–reheating thermal background.

6.2.6 Late decays and photodissociation/hadrodissociation gates

Any long–lived relic X that decays electromagnetically or hadronically after BBN injects entropy and can spoil light–element yields. A conservative two–limb gate is:

$$\textbf{(L1)}: \quad \tau_X \lesssim 0.1\ \text{s} \quad \text{or} \quad \textbf{(L2)}: \quad \zeta_X \equiv m_X\,Y_X \lesssim \zeta_{\rm crit}(\tau_X), \tag{6.28}$$

where $\zeta_{\rm crit}(\tau)$ is the critical injected energy per photon allowed by BBN/CMB bounds (rapidly decreasing for $\tau \sim 10^2 - 10^8$ s). In practice, either decays occur before BBN or the abundance must be exponentially suppressed.

6.2.7 Summary of reheating consistency gates

We collect the minimal set to be enforced in scans:

(R1) Thermalization: $T_{\rm rh} \gtrsim 5$ MeV, Eq. (6.17),

(R2) Gravity drift: $\left|M_{\rm eff}(T_{\rm BBN}) - M_{\rm Pl}\right|/M_{\rm Pl} \lesssim \mathcal{O}(10^{-1})$, Eq. (6.22),

(R3) Extra radiation: $\Delta N_{\rm eff}(T_{\rm BBN/CMB})$ from f_S below observational bound, Eq. (6.24),

(R4) Relics: $\Omega_{\rm relic}^{\rm tot}h^2 \le 0.12$ and L1/L2 in Eq. (6.28),

(R5) Restoration: post–reheating propagation $c_T \to 1$, $c_s(\theta) \to c_s^{\rm iso}$, with negligible residual anisotropy.

Outlook. Equations (6.17)–(6.28) provide the map from microscopic spectral parameters $(\Gamma_\eta, g_{\eta\Phi}, h_{\eta S}, \kappa_\eta, M_{\rm eff})$ to macroscopic early–thermal conditions. In Sec. 6.3 we propagate these initial conditions into linear transfer functions that seed large–scale structure with distinctive spectral imprints.

6.3 Hand–off to Linear Growth and Initial Conditions for LSS

With reheating complete and the visible sector thermalized at $T_{\rm rh}$ (Sec. 6.2), the primordial perturbations generated during the η–wave phase (Sec. 4) specify the *initial Cauchy data* for the linear Einstein–Boltzmann evolution of photons, baryons, neutrinos, and cold dark matter. In this section we (i) define the matching hypersurface from the inflationary to the radiation era, (ii) translate the conserved curvature mode ζ (with mild statistical anisotropy) into species perturbations, (iii) state the transfer operators that propagate these to CMB/LSS observables, and (iv) collect growth–equation and consistency gates ensuring restoration to GR–like propagation.

6.3.1 Matching hypersurface and conserved curvature

We choose the end–of–reheating hypersurface $\Sigma_{\rm rh}$ defined by

$$H(t_{\rm rh}) \simeq \Gamma_\eta^{\rm eff}(t_{\rm rh}), \qquad \rho_R(t_{\rm rh}) = \frac{\pi^2}{30} g_*(T_{\rm rh}) T_{\rm rh}^4, \qquad u^\mu \text{ aether rest frame}, \tag{6.29}$$

and work in the *comoving–aether gauge* (Vol. I): $\delta T = 0$, $u^i = 0$ on $\Sigma_{\rm rh}$. On super–Hubble scales, the gauge–invariant curvature perturbation on uniform–density slices,

$$\zeta \;\equiv\; -\psi \;-\; \frac{H}{\dot\rho_{\rm tot}}\,\delta\rho_{\rm tot}, \tag{6.30}$$

is conserved in the absence of entropy production between species.[1] Statistical anisotropy from the director $\hat\chi^{(0)}$ is parameterized as

$$\mathcal{P}_\zeta(\mathbf{k}) \;=\; \mathcal{P}_0(k)\left[1 + g_*\,(\hat{\mathbf{k}}\cdot\hat\chi^{(0)})^2\right], \qquad |g_*| \ll 1, \tag{6.31}$$

with $\mathcal{P}_0(k)$ given by Eq. (V2. 4.5) and g_* computed from the small difference $c_\parallel - c_\perp$ during inflation.

6.3.2 Adiabatic initial conditions and isocurvature suppression

Under adiabaticity, the relative entropy between any two fluids vanishes on $\Sigma_{\rm rh}$:

$$S_{ij} \;\equiv\; \frac{\delta\rho_i}{\rho_i + p_i} \;-\; \frac{\delta\rho_j}{\rho_j + p_j} \;=\; 0, \qquad \forall\, i,j \in \{\gamma,\nu,b,{\rm cdm}\}. \tag{6.32}$$

Equivalently, each species shares the same *curvature* perturbation $\zeta_i = \zeta$. In Newtonian gauge ($ds^2 = -(1+2\Phi)dt^2 + a^2(1-2\Psi)d\mathbf{x}^2$) and for $k \ll aH$,

$$\delta_i \;\equiv\; \frac{\delta\rho_i}{\rho_i} \;=\; -3(1+w_i)\left(\zeta + \frac{\dot\Psi}{H}\right) \;\xrightarrow{k\ll aH}\; -3(1+w_i)\,\zeta, \qquad \theta_i \equiv \nabla\cdot\mathbf{v}_i \;=\; \mathcal{O}(k^2), \tag{6.33}$$

to leading order. Thus on $\Sigma_{\rm rh}$ we may set

$$\delta_\gamma = \delta_\nu = -4\,\zeta, \qquad \delta_b = \delta_{\rm cdm} = -3\,\zeta, \qquad \theta_i = 0, \qquad \Phi = \Psi = \Phi_{\rm rh}(\zeta), \tag{6.34}$$

with $\Phi_{\rm rh}$ fixed by the Hamiltonian constraint and by the (small) anisotropic stress of free–streaming neutrinos.[2] Any η–sourced isocurvature mode is constrained by CMB to be subdominant; in practice we demand

$$\alpha_{\rm iso} \;\equiv\; \frac{\mathcal{P}_S}{\mathcal{P}_\zeta + \mathcal{P}_S}\Big|_{k_\star} \;\lesssim\; 10^{-2}, \qquad \text{for } S \in \{S_{b\gamma}, S_{{\rm cdm}\,\gamma}, S_{\nu\gamma}\}. \tag{6.35}$$

6.3.3 Transfer operators and observables

Linear evolution is conveniently written using transfer kernels T that map the primordial ζ (and tensor mode h) to late–time fields:

$$\delta_m(\mathbf{k},z) \;=\; \mathsf{T}_m(k,z;\hat\chi^{(0)})\,\zeta(\mathbf{k}) \quad\Rightarrow\quad P_m(\mathbf{k},z) \;=\; |\mathsf{T}_m|^2\,\mathcal{P}_\zeta(\mathbf{k}), \tag{6.36}$$

$$\Delta_\ell^X(k) \;=\; \mathsf{T}_\ell^X(k;\hat\chi^{(0)})\,\zeta(\mathbf{k}), \quad X \in \{T,E,B\}, \quad\Rightarrow\quad C_\ell^{XY} \;=\; 4\pi\int\frac{dk}{k}\,\mathcal{P}_\zeta(k)\,\Delta_\ell^X(k)\Delta_\ell^Y(k), \tag{6.37}$$

where the mild anisotropy enters as a *quadrupolar* modulation of the radiation and matter kernels:

$$\mathsf{T}(k;\hat\chi^{(0)}) \;=\; \mathsf{T}_0(k)\left[1 + \tfrac{1}{2}\,\tilde g_*(k)\,(\hat{\mathbf{k}}\cdot\hat\chi^{(0)})^2\right], \qquad |\tilde g_*(k)| \sim \mathcal{O}(g_*). \tag{6.38}$$

In the late matter era (restoration gate satisfied), the standard relation

$$P_m(k,z) \;=\; \left[\frac{2}{5}\,\frac{k^2\,T_\Phi(k)\,D(z)}{\Omega_{m0}H_0^2}\right]^2 \mathcal{P}_\zeta(k) \tag{6.39}$$

holds, where $T_\Phi(k)$ is the potential transfer function and $D(z)$ the linear growth factor (normalized to $D(0) = 1$). Spectral imprints modify $T_\Phi(k)$ only through the initial conditions (acoustic phase, diffusion scale) and through small $\mathcal{O}(g_*)$ angular modulations.

[1] Residual dissipation from the η sector after $t_{\rm rh}$ must be negligible; this is our Restoration Gate in Sec. 6.2.

[2] At later times, $\Psi \neq \Phi$ at the few–percent level due to neutrino anisotropic stress; the difference inherits the same g_* angular pattern if $g_* \neq 0$.

6.3.4 Boltzmann hierarchy with spectral imprints

The photon and neutrino multipoles obey the standard hierarchies with *isochronous* initial conditions sourced by ζ. The only model–specific inputs are:

1. **Acoustic scale.** The photon–baryon sound speed $c_{s,\gamma b}^2 = 1/\big[3(1+R)\big]$ with $R \equiv 3\rho_b/4\rho_\gamma$ is standard. The *phase* of acoustic oscillations inherits the primordial ζ statistics including the g_* quadrupole.

2. **Diffusion (Silk) scale.** The diffusion wavenumber is

$$k_D^{-2}(\tau) \;=\; \frac{1}{6}\int^{\tau} d\tau' \, \frac{1}{\dot\kappa(\tau')} \, \frac{R^2 + \frac{16}{15}(1+R)}{(1+R)^2}, \qquad \dot\kappa = a\, n_e\, \sigma_T, \tag{6.40}$$

   unmodified at leading order; spectral corrections enter only via the initial ζ and g_*.

3. **Tensor sector.** Gravitational waves obey $c_T^2 = 1 + \delta c_T(\mathcal{S})$ during inflation but must restore to $c_T \to 1$ by recombination. We impose $|\delta c_T(z_*)| \ll 10^{-15}$ (multimessenger bound translated to recombination).

6.3.5 Growth factor with state–dependent Planck mass

If $M_{\rm eff}$ has fully restored to $M_{\rm Pl}$ by equality, the growth obeys

$$D''(a) + \Big[\frac{3}{a} + \frac{d\ln H}{d\ln a}\frac{1}{a}\Big] D'(a) \;-\; \frac{3}{2}\,\frac{\Omega_m(a)}{a^2}\, D(a) \;=\; 0, \tag{6.41}$$

with primes $' \equiv d/da$. A small drift $M_{\rm eff}(a) = M_{\rm Pl}(1+\epsilon_M)$ modifies the source by a factor $\mu(a) \equiv M_{\rm Pl}^2/M_{\rm eff}^2 \simeq 1 - 2\epsilon_M$, giving

$$D'' + \Big[\frac{3}{a} + \frac{d\ln H}{d\ln a}\frac{1}{a}\Big] D' - \frac{3}{2}\,\mu(a)\,\frac{\Omega_m(a)}{a^2}\, D = 0, \quad |\mu(a) - 1| \ll 1 \text{ (restoration gate)}. \tag{6.42}$$

This feeds directly into σ_8 and redshift–space distortions $f\sigma_8$.

6.3.6 BAO, phase shifts, and anisotropy diagnostics

Baryon Acoustic Oscillations are described by

$$P_{gg}(\mathbf{k}, z) \;=\; b_1^2(z)\, P_m(\mathbf{k}, z)\, \Big[1 + A_{\rm BAO}(k)\, \sin(2\pi k\, r_s + \varphi_{\rm spec})\Big], \tag{6.43}$$

where r_s is the standard sound horizon at drag epoch, while $\varphi_{\rm spec}$ is a small *primordial* phase inherited from the inflationary $c_s(\theta)$ dependence and possible $g_* \neq 0$. A conservative diagnostic set is:

(D1) BAO phase stability: $|\varphi_{\rm spec}| \ll 1$,

(D2) Quadrupolar modulation: $P_m(\mathbf{k}) = P_0(k)\,[1 + \mathcal{Q}(k)\,(\hat{\mathbf{k}}\cdot\hat\chi^{(0)})^2]$, $|\mathcal{Q}(k)| \sim \mathcal{O}(g_*)$,

(D3) Consistency with CMB anisotropy bound: $|g_*|$ below Planck/next–gen limits.

6.3.7 Implementation recipe (CLASS/CAMB–style)

For practical scans:

1. Input $\mathcal{P}_\zeta(k, \hat{\mathbf{k}})$ as in Eq. (6.31) with (n_s, r, g_*) from Chap. 4.

2. Set adiabatic super–Hubble IC on $\Sigma_{\rm rh}$ via Eqs. (6.33)–(6.34).

3. Ensure $c_T \to 1$ and $M_{\rm eff} \to M_{\rm Pl}$ by $z \gtrsim 10^5$; else include $\mu(a)$ in Eq. (6.42).

4. Evolve the Einstein–Boltzmann system; extract C_ℓ and $P_m(k, z)$ with anisotropy diagnostics (D1–D3).

6.3.8 Consistency gates for the hand–off

- **(H1) Restoration:** $c_T(z_*) \simeq 1,\ \mu(z_*) \simeq 1$ at recombination.
- **(H2) Adiabaticity:** $\alpha_{\rm iso} \lesssim 10^{-2}$ at $k_\star$ (Eq. (6.35)).
- **(H3) Linear causality:** All sound speeds $c_s \leq 1$; no superluminal group velocities under $g_{\rm eff}$.
- **(H4) Anisotropy bounds:** $|g_*|$ and $|\mathcal{Q}(k)|$ within CMB/LSS limits.

Summary. Eqs. (6.29)–(6.39) provide the clean relay from spectral–viscosity inflation to standard linear cosmology. All model dependence is concentrated in the primordial spectrum (including a calculable g_*) and in tiny, bounded restoration drifts (c_T, μ). The resulting pipeline is thus predictive yet conservative, ready for direct confrontation with CMB and LSS data.

Chapter 7

Cosmic Web as a Spectral Resonance Pattern

7.1 Interference of Unstable Modes and Chladni-like Morphology

The cosmic web is here interpreted as a macroscopic *interference (resonance) pattern* of the most unstable spectral modes excited during the loss of stability of the Quiescent Void (Sec. **??**). In this section we construct the interference field, derive its anisotropic k–space support from the linear growth operator, and formulate ridge/node conditions (Morse–theoretic skeleton) that underlie filaments, walls, and voids. The result is a "cosmic Chladni figure": a pattern of nodal sheets and ridge filaments selected by the dispersion anisotropy and weak nonlinear saturation.

7.1.1 Linear growth kernel and the unstable band

Let $\delta\mathbf{S}$ denote the vector of perturbations about S_0 (Vol. I), governed at quadratic order by

$$\partial_t\,\delta\mathbf{S} \;=\; \mathsf{A}\,\delta\mathbf{S} \;+\; \mathsf{B}^{ij}\,\partial_i\partial_j\delta\mathbf{S} \;+\; \cdots, \qquad \text{cf. Eq. (V2. 2.1).} \tag{7.1}$$

For plane waves $\propto e^{i\mathbf{k}\cdot\mathbf{x}+\lambda t}$ one obtains the spectral problem

$$\det\Big[\lambda\mathbb{1} - \mathsf{A} + \mathsf{B}^{ij}k_ik_j\Big] = 0, \qquad \lambda_\alpha(\mathbf{k}) \;=\; \lambda_{\alpha,0} \;-\; D_{\alpha,\parallel}\,k_\parallel^2 \;-\; D_{\alpha,\perp}\,k_\perp^2 + \mathcal{O}(k^4), \tag{7.2}$$

with $k_\parallel \equiv \mathbf{k}\cdot\hat{\chi}^{(0)}$, $k_\perp^2 = k^2 - k_\parallel^2$, and diffusivities $D_{\alpha,\parallel/\perp} > 0$ inherited from the principal symbol under $g_{\rm eff}$. The *most unstable band* $\mathcal{B}$ is the set of $\mathbf{k}$ maximizing $\lambda_{\max}(\mathbf{k})$:

$$\mathcal{B} \;=\; \Big\{\mathbf{k} \;:\; \lambda_{\max}(\mathbf{k}) = \lambda_{\rm peak} - \tfrac{1}{2}(\mathbf{k}-\mathbf{k}_\star)^\top\,\mathsf{C}\,(\mathbf{k}-\mathbf{k}_\star) + \cdots\Big\}, \tag{7.3}$$

where C is a positive matrix (curvature in k–space) and $\mathbf{k}_\star$ can be anisotropically distributed on an ellipsoidal shell:

$$\frac{k_\parallel^2}{k_{\star,\parallel}^2} \;+\; \frac{k_\perp^2}{k_{\star,\perp}^2} \;\simeq\; 1, \qquad k_{\star,\parallel/\perp}^2 \;\equiv\; \frac{\lambda_{\rm peak}}{D_{\max,\parallel/\perp}}. \tag{7.4}$$

The correlation length at onset is $\xi_{\parallel/\perp} \equiv k_{\star,\parallel/\perp}^{-1}$.

7.1.2 Interference field and structure factor

We compress the multi–component perturbations into a scalar *pattern field* $\mathcal{I}(x)$ that seeds density/curvature modulations,[1] built as a superposition of modes in $\mathcal{B}$:

$$\mathcal{I}(\mathbf{x},t) \;=\; \mathrm{Re}\int_{\mathcal{B}}\frac{d^3k}{(2\pi)^3}\,A(\mathbf{k})\,e^{i\mathbf{k}\cdot\mathbf{x}}\,\exp\Big[\int_{t_0}^{t}\lambda_{\max}(\mathbf{k},t')\,dt'\Big], \tag{7.5}$$

[1] E.g. in comoving–aether gauge one may choose $\mathcal{I}$ to be a canonically normalized linear combination projecting $\delta\mathbf{S}$ onto the growing scalar channel that sources ζ; cf. Eq. (V2. 4.4).

with random phases absorbed in $A(\mathbf{k})$. The structure factor is

$$S(\mathbf{k},t) \;\equiv\; \langle|\tilde{\mathcal{I}}(\mathbf{k},t)|^2\rangle \;=\; |A(\mathbf{k})|^2\,\exp\Big[2\int_{t_0}^{t}\lambda_{\max}(\mathbf{k},t')\,dt'\Big], \tag{7.6}$$

peaked on the ellipsoid (7.4). The two–point correlation

$$C(\mathbf{r}) \;\equiv\; \langle\mathcal{I}(\mathbf{x})\,\mathcal{I}(\mathbf{x}+\mathbf{r})\rangle = \int\frac{d^3k}{(2\pi)^3}\,S(\mathbf{k})\,e^{i\mathbf{k}\cdot\mathbf{r}} \tag{7.7}$$

decays anisotropically with $r_\parallel/\xi_\parallel$ and $r_\perp/\xi_\perp$, encoding an intrinsic directional memory of $\hat{\chi}^{(0)}$.

7.1.3 Amplitude equations near threshold (pattern selection)

Close to onset ($\lambda_{\rm peak} \gtrsim 0$) the slowly varying complex envelopes $A_n(\mathbf{X},T)$ for a finite set of nearly degenerate wavevectors $\{\mathbf{k}_n\}\subset\mathcal{B}$ obey coupled Ginzburg–Landau equations (obtained by multiscale expansion of the full field equations):

$$\partial_T A_n \;=\; \mu\,A_n \;+\; \sum_{i,j} d_{ij}^{(n)}\,\partial_i\partial_j A_n \;-\; \sum_m g_{nm}\,|A_m|^2\,A_n \;-\; \sum_{m\neq\ell} h_{nm\ell}\,A_m A_\ell \bar{A}_p\,\delta_{\mathbf{k}_m+\mathbf{k}_\ell-\mathbf{k}_p,\mathbf{k}_n} \;+\;\cdots, \tag{7.8}$$

with growth parameter $\mu\propto\lambda_{\rm peak}$, anisotropic diffusion $d_{ij}^{(n)}$ inherited from $D_{\parallel/\perp}$, and cubic couplings (g,h) fixed by the unified Lagrangian (Vol. I, Ch. 3). The resonant triads (last term) lock phases and select planform:

- *Stripe/lamellar* patterns for narrow $\mathcal{B}$: one dominant $\mathbf{k}_\star$ yields sheets (walls).
- *Hexagonal/tubular* patterns for near–isotropic shells: three–mode resonances generate filamentary lattices.
- *Anisotropic filaments* when $d_{ij}^{(n)}$ favors propagation along $\hat{\chi}^{(0)}$.

Weak nonlinearity saturates growth at $|A_n|^2\sim\mu/g_{nn}$, setting the web contrast before gravitational amplification.

7.1.4 Ridge, wall, and void conditions (Morse–skeleton)

Let $H_{ij}(\mathbf{x})\equiv\partial_i\partial_j\mathcal{I}$ be the Hessian and $(\lambda_1\geq\lambda_2\geq\lambda_3,\ \mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ its eigenpairs. The *ridge (filament) set* $\mathcal{R}$ is the locus where (locally) $\mathcal{I}$ is maximized transversely:

$$\nabla\mathcal{I}\;\parallel\;\mathbf{e}_1,\qquad \lambda_2<0,\ \lambda_3<0, \tag{7.9}$$

while *walls* correspond to one negative transverse curvature:

$$\nabla\mathcal{I}\;\parallel\;\mathbf{e}_1,\qquad \lambda_2<0,\ \lambda_3\approx 0\ \text{(or weakly negative)}. \tag{7.10}$$

Voids are basins of attraction of the gradient flow $\dot{\mathbf{x}}=\nabla\mathcal{I}$, typically near minima ($\lambda_3>0$). The *skeleton* is the network of integral curves of $\mathbf{e}_1$ connecting critical points of decreasing Morse index; numerically, one extracts $\mathcal{R}$ from the conditions

$$\mathbf{e}_2\cdot\nabla\mathcal{I}=0=\mathbf{e}_3\cdot\nabla\mathcal{I},\qquad \lambda_{2,3}<0, \tag{7.11}$$

on a smoothed $\mathcal{I}$ field (see Sec. 7.2 for statistics).

7.1.5 Anisotropy diagnostics and "Chladni" analogy

The directional imprint of $\hat{\chi}^{(0)}$ is captured by the gradient–quadrupole tensor

$$\mathsf{Q}_{ij} \;\equiv\; \langle\partial_i\mathcal{I}\,\partial_j\mathcal{I}\rangle \;-\; \tfrac{1}{3}\,\delta_{ij}\,\langle|\nabla\mathcal{I}|^2\rangle,\qquad \mathcal{A}\;\equiv\;\frac{\lambda_{\max}(\mathsf{Q})-\lambda_{\min}(\mathsf{Q})}{\operatorname{tr}\mathsf{Q}}, \tag{7.12}$$

with $\mathcal{A}\sim\mathcal{O}(g_*)$ at onset. The Chladni analogy is precise: $\mathcal{I}$ plays the role of a vibration profile on a resonant "drum" whose effective boundary is the finite coherence domain set by the η–wave and by the horizon at the time of excitation. Nodal sets $\mathcal{I}=0$ form sheets; their intersections define filaments; basins define voids.

7.1.6 From pattern field to density seeds

To leading order we map $\mathcal{I}$ to initial matter contrast by a local bias expansion,

$$\delta\rho/\bar{\rho} \;\equiv\; \delta_m \;=\; \alpha_1\,\mathcal{I} \;+\; \alpha_2\,\mathcal{I}^2 \;+\; \beta_1\,\nabla^2\mathcal{I} \;+\; \cdots\,, \tag{7.13}$$

where (α_i, β_i) are calculable from matching the growing scalar channel to the matter sector at the end of reheating (Sec. 6.3). This mapping preserves the ridge/wall topology at large scales and feeds the web seed into the GR linear regime.

Summary. Equations (7.2)–(7.13) endow the cosmic web with a first–principles origin as a spectral resonance pattern: an ellipsoidal band of unstable modes interferes to create a filament–wall–void mosaic (Chladni–like), weakly anisotropic along the internal director $\hat{\chi}^{(0)}$, and subsequently amplified by gravity. The next section quantifies this morphology via Minkowski functionals, skeleton statistics, and higher–order spectra beyond pure gravitational predictions.

7.2 Morphological Statistics Beyond Pure Gravity

If the web is seeded by the spectral interference field $\mathcal{I}$ (Sec. 7.1) rather than by purely gravitational amplification of a statistically isotropic Gaussian potential, then its *morphological statistics* must encode this origin. In this section we (i) define a suite of shape diagnostics in configuration space (Minkowski functionals, Betti numbers, skeleton length, shapefinders), (ii) connect their amplitudes to the k–space structure factor $S(\mathbf{k})$ of Eq. (7.6), including anisotropy with respect to the internal director $\hat{\chi}^{(0)}$, and (iii) introduce higher–order Fourier diagnostics (bispectrum/trispectrum) with characteristic angular dependence that distinguishes spectral resonance from pure–gravity ΛCDM templates.

7.2.1 Gaussian reference and spectral moments

Let the *smoothed* pattern field be

$$\mathcal{I}_R(\mathbf{x}) \;=\; \int d^3y\, W_R(|\mathbf{x}-\mathbf{y}|)\,\mathcal{I}(\mathbf{y}), \qquad \widetilde{W}_R(\mathbf{k}) = \exp\big(-k^2R^2/2\big) \;\text{ for concreteness,} \tag{7.14}$$

and define anisotropic spectral moments with respect to the director $\hat{\chi}^{(0)}$:

$$\sigma_0^2(R) \equiv \langle \mathcal{I}_R^2\rangle \;=\; \int \frac{d^3k}{(2\pi)^3}\, S(\mathbf{k})\,\widetilde{W}_R^2(\mathbf{k}), \tag{7.15}$$

$$\sigma_{1,\parallel}^2(R) \equiv \langle |\nabla_\parallel \mathcal{I}_R|^2\rangle \;=\; \int \frac{d^3k}{(2\pi)^3}\, k_\parallel^2\, S(\mathbf{k})\,\widetilde{W}_R^2(\mathbf{k}), \tag{7.16}$$

$$\sigma_{1,\perp}^2(R) \equiv \tfrac{1}{2}\langle |\nabla_\perp \mathcal{I}_R|^2\rangle \;=\; \tfrac{1}{2}\int \frac{d^3k}{(2\pi)^3}\, k_\perp^2\, S(\mathbf{k})\,\widetilde{W}_R^2(\mathbf{k}), \tag{7.17}$$

with $k_\parallel \equiv \mathbf{k}\cdot\hat{\chi}^{(0)}$ and $k_\perp^2 = k^2 - k_\parallel^2$. An anisotropy estimator is

$$\mathcal{A}_\nabla(R) \;\equiv\; \frac{\sigma_{1,\parallel}^2 - \sigma_{1,\perp}^2}{\sigma_{1,\parallel}^2 + 2\sigma_{1,\perp}^2}\,, \qquad \mathcal{A}_\nabla = 0 \;\;\text{(isotropy).} \tag{7.18}$$

7.2.2 Minkowski functionals and Gaussian baselines

For the excursion set $\mathcal{Q}_\nu \equiv \{\mathbf{x}:\; \nu \equiv \mathcal{I}_R/\sigma_0 \geq \nu_{\rm thr}\}$, the (intensive) Minkowski functionals in 3D are

$$\mathcal{V}_0(\nu) = \frac{1}{V}\int_{\mathcal{Q}_\nu} d^3x, \quad \mathcal{V}_1(\nu) = \frac{1}{6V}\int_{\partial\mathcal{Q}_\nu} dS, \quad \mathcal{V}_2(\nu) = \frac{1}{6\pi V}\int_{\partial\mathcal{Q}_\nu} H\,dS, \quad \mathcal{V}_3(\nu) = \frac{1}{4\pi V}\int_{\partial\mathcal{Q}_\nu} K\,dS, \tag{7.19}$$

where H (K) is the mean (Gaussian) curvature of the boundary. For a *Gaussian, statistically isotropic* field these are (up to known constants) functions of ν and of the ratio σ_1/σ_0 only. We write the baseline compactly as

$$\mathcal{V}_\mu^{(\rm G)}(\nu; R) \;=\; \mathcal{C}_\mu(R)\, e^{-\nu^2/2}\,\mathcal{H}_{n_\mu}(\nu) \quad (\mu = 1,2,3), \tag{7.20}$$

where $\mathcal{H}_n$ are Hermite polynomials and the amplitudes $\mathcal{C}_\mu$ depend on σ_0, σ_1 (and on the smoothing kernel). In our case, *anisotropy* modifies $\mathcal{C}_\mu$ by replacing $\sigma_1^2 \to \frac{1}{3}(\sigma_{1,\parallel}^2 + 2\sigma_{1,\perp}^2)$ at leading order and induces subleading angular dependence (see below).

Weak *non-Gaussianity* (from mode–coupling of the resonance envelopes, Eq. (7.8)) produces controlled corrections via the Edgeworth expansion:

$$\mathcal{V}_\mu(\nu) = \mathcal{V}_\mu^{(\mathrm{G})}(\nu) + \Delta\mathcal{V}_\mu(\nu), \qquad \Delta\mathcal{V}_\mu \;=\; \sum_p \mathcal{S}_p(R)\, \mathcal{F}_p^{(\mu)}(\nu), \tag{7.21}$$

where the skewness–like cumulants $\mathcal{S}_p$ are integrals of the bispectrum of $\mathcal{I}_R$ (Sec. **??**). The shapes $\mathcal{F}_p^{(\mu)}$ are fixed polynomials in ν times $e^{-\nu^2/2}$.

Filamentarity and planarity (shapefinders). From $(\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3)$ one defines *shapefinders* [**?**]:

$$T \equiv \frac{3\mathcal{V}_0}{\mathcal{V}_1}, \qquad B \equiv \frac{\mathcal{V}_1}{\mathcal{V}_2}, \qquad L \equiv \frac{\mathcal{V}_2}{4\pi\mathcal{V}_3}, \qquad \text{Planarity} \;=\; \frac{B-T}{B+T}, \quad \text{Filamentarity} \;=\; \frac{L-B}{L+B}, \tag{7.22}$$

which peak for walls and filaments, respectively. Spectral resonance predicts an *enhanced filamentarity* at thresholds $\nu \sim \mathcal{O}(1)$ aligned with $\hat{\chi}^{(0)}$ when the ellipsoidal band (7.4) has $k_{\star,\parallel} \neq k_{\star,\perp}$.

7.2.3 Topology: Euler characteristic and Betti numbers

The Euler characteristic density $\chi \equiv \mathcal{V}_3$ relates to the Betti numbers $(\beta_0, \beta_1, \beta_2)$ of $\mathcal{Q}_\nu$:

$$\chi(\nu) \;=\; \frac{\beta_0 - \beta_1 + \beta_2}{V}. \tag{7.23}$$

In an isotropic Gaussian model, $\chi(\nu)$ is an odd function with a universal S–shape. Spectral anisotropy induces (i) a *shift* of the zero–crossing ν_χ through $\mathcal{C}_\mu(R)$ and (ii) a *broadening* proportional to $\mathcal{A}_\nabla(R)$, reflecting preferential ridge alignment. We quantify topology via the *genus* $g = -\chi/2$ and track the *percolation threshold* ν_{perc} at which β_0 (or β_2) shows a sharp change; spectral resonance typically lowers ν_{perc} for the filamentary phase relative to ΛCDM with the same $P(k)$.

7.2.4 Skeleton length and ridge statistics

Consider the eigenpairs $(\lambda_i, \mathbf{e}_i)$ of the Hessian $H_{ij} = \partial_i\partial_j\mathcal{I}_R$. The *ridge set* satisfies Eq. (7.11). A practical estimator for the *skeleton length density* $\mathcal{L}_{\mathrm{skel}}$ at threshold ν is

$$\mathcal{L}_{\mathrm{skel}}(\nu; R) \;=\; \left\langle \delta\!\big(\mathbf{e}_2\cdot\nabla\mathcal{I}_R\big)\, \delta\!\big(\mathbf{e}_3\cdot\nabla\mathcal{I}_R\big)\, |\lambda_2\lambda_3|\, \Theta(-\lambda_2)\Theta(-\lambda_3)\, \Theta(\mathcal{I}_R - \nu\sigma_0)\right\rangle, \tag{7.24}$$

which for a (weakly) Gaussian field reduces to a closed form involving $(\sigma_0, \sigma_{1,\parallel}, \sigma_{1,\perp})$ and spectral kurtoses. After angular averaging relative to $\hat{\chi}^{(0)}$, the leading anisotropy enters as

$$\mathcal{L}_{\mathrm{skel}}(\nu; R, \theta) \;=\; \mathcal{L}_0(\nu; R)\left[1 + \gamma_{\mathrm{skel}}(R)\, P_2(\cos\theta) + \cdots\right], \qquad \gamma_{\mathrm{skel}} \;\propto\; \mathcal{A}_\nabla(R), \tag{7.25}$$

with P_2 the Legendre polynomial and $\theta = \angle(\mathbf{e}_1, \hat{\chi}^{(0)})$. The coefficient γ_{skel} is a *direct, real–space* probe of the director–induced anisotropy.

7.2.5 Bispectrum, trispectrum, and anisotropic shapes

Let δ_m denote the initial matter contrast mapped from $\mathcal{I}$ via Eq. (7.13). The bispectrum is

$$\langle \tilde{\delta}_m(\mathbf{k}_1)\tilde{\delta}_m(\mathbf{k}_2)\tilde{\delta}_m(\mathbf{k}_3)\rangle \;=\; (2\pi)^3\delta_D(\mathbf{k}_{123})\, B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3), \tag{7.26}$$

and in the spectral–resonance scenario admits the decomposition

$$B \;=\; B_{\mathrm{iso}}(k_1, k_2, k_3) \;+\; g_*(R) \sum_{i=1}^{3} \mathcal{G}(k_i, k_j, k_\ell)\left[\hat{\mathbf{k}}_i\cdot\hat{\chi}^{(0)}\right]^2 \;+\; \cdots, \tag{7.27}$$

where $B_{\rm iso}$ is the isotropic template (equilateral–like for local envelope mode–coupling) and $\mathcal{G}$ encodes the k–dependence from the amplitude system (7.8). The angle–averaged skewness that enters Eq. (7.21) reads

$$\mathcal{S}_0(R) \;=\; \frac{1}{\sigma_0^4}\int \frac{d^3k_1 d^3k_2}{(2\pi)^6}\,\widetilde{W}_R(k_1)\widetilde{W}_R(k_2)\widetilde{W}_R(k_{12})\,B(\mathbf{k}_1,\mathbf{k}_2,-\mathbf{k}_{12}), \tag{7.28}$$

with anisotropic corrections $\propto g_*$ calculable analytically for Gaussian windows. A complementary diagnostic is the *line correlation function* $\ell(r)$ (a three–point phase statistic) which is especially sensitive to filamentarity; in our case,

$$\ell(r,\theta) \;=\; \ell_0(r)\Big[1+\gamma_\ell(r)\,P_2(\cos\theta)+\cdots\Big], \qquad \gamma_\ell(r) \;\propto\; \mathcal{A}_\nabla(R\sim r), \tag{7.29}$$

again providing a clean angular handle on resonance–induced anisotropy.

7.2.6 Discriminants against ΛCDM with matched $P(k)$

A central virtue of morphology–based diagnostics is that they can distinguish two models with the *same* isotropic power spectrum:

1. **Anisotropy ratios:** $\mathcal{A}_\nabla(R)$ from Eq. (7.18), the skeleton anisotropy $\gamma_{\rm skel}(R)$, and the bispectrum coefficient g_* in Eq. (7.27) are jointly nonzero in the spectral resonance but vanish (within errors) for statistically isotropic ΛCDM ICs.

2. **Genus and percolation:** the shift of $(\nu_\chi,\nu_{\rm perc})$ at fixed $P(k)$ reflects coherent planform selection (Eq. (7.8)) rather than random–phase Gaussianity.

3. **Shapefinders:** enhanced filamentarity at moderate thresholds and its alignment with $\hat{\chi}^{(0)}$ cannot be reproduced by changing $P(k)$ alone.

7.2.7 Practical estimators and error control

For survey data (or mocks) one proceeds as follows:

- Smooth the density field to a set of scales R; compute $(\sigma_0,\sigma_{1,\parallel},\sigma_{1,\perp})$ and $\mathcal{A}_\nabla(R)$.
- Measure $(\mathcal{V}_\mu(\nu),\ \beta_{0,1,2}(\nu))$ and fit $\Delta\mathcal{V}_\mu$ using Eq. (7.21) to extract $\mathcal{S}_p$ and g_*.
- Extract the ridge skeleton and its length density $\mathcal{L}_{\rm skel}(\nu)$; fit the angular modulation $\gamma_{\rm skel}$ from Eq. (7.25).
- Estimate $B(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$ in angle–binned triangles relative to $\hat{\chi}^{(0)}$ and constrain g_* jointly with morphology.

Systematics (survey mask, RSD, bias) are controlled by forward models and null tests on isotropic mocks with the same $P(k)$ and selection functions.

Summary. Morphological statistics elevate the spectral–resonance picture into falsifiable predictions: anisotropic gradients, shifted genus and percolation thresholds, enhanced filamentarity aligned with $\hat{\chi}^{(0)}$, and angular bispectrum modulations. None of these arise in a purely Gaussian, statistically isotropic ΛCDM initial state with matched power, providing sharp discriminants for forthcoming LSS datasets.

7.3 Comparison with N-body and Semi-Analytic Models

We now describe how to confront the spectral–resonance framework with late–time structure formation tools. The central task is to embed the *interference origin* of the web (Sec. 7.1)—including its controlled anisotropy relative to $\hat{\chi}^{(0)}$—into initial conditions (ICs), evolve them with standard N–body pipelines (or semi–analytic LPT), and read out discriminants that cannot be mimicked by an isotropic Gaussian ΛCDM field with matched $P(k)$.

7.3.1 IC synthesis with resonance and anisotropy

Let $\delta_{\rm G}(\mathbf{k})$ be a zero–mean Gaussian field with isotropic spectrum $P_{\rm iso}(k)$, $\langle\delta_{\rm G}(\mathbf{k})\delta_{\rm G}(\mathbf{k}')\rangle = (2\pi)^3\delta_D(\mathbf{k}+\mathbf{k}')P_{\rm iso}(k)$. We construct an *augmented* IC field

$$\delta_{\rm IC}(\mathbf{k}) \;=\; \mathcal{T}(k,\mu)\,\delta_{\rm G}(\mathbf{k}) \;+\; \int\frac{d^3q}{(2\pi)^3}\,\mathcal{M}(\mathbf{k};\mathbf{q})\,\delta_{\rm G}(\mathbf{q})\,\delta_{\rm G}(\mathbf{k}-\mathbf{q}), \qquad \mu\equiv\hat{\mathbf{k}}\cdot\hat{\chi}^{(0)}, \tag{7.30}$$

with two ingredients:

- An *anisotropic transfer* $\mathcal{T}(k,\mu)$ encoding the interference band and spectral diffusivities,

$$\mathcal{T}(k,\mu) \;=\; \sqrt{\frac{S(\mathbf{k})}{P_{\rm iso}(k)}} \;=\; \exp\left[-\frac{(k_\parallel-k_{\star,\parallel})^2}{2\Sigma_\parallel^2}-\frac{(k_\perp-k_{\star,\perp})^2}{2\Sigma_\perp^2}\right]\times\mathcal{N}(k), \tag{7.31}$$

  where $S(\mathbf{k})$ is the structure factor of Eq. (7.6), $k_\parallel\equiv\mathbf{k}\cdot\hat{\chi}^{(0)}$, $k_\perp^2=k^2-k_\parallel^2$, and $\mathcal{N}(k)$ fixes the isotropic large–scale normalization.

- A *phase–coupling kernel* $\mathcal{M}$ to realize the bispectrum (and trispectrum) of Sec. **??**. A minimal choice that reproduces the anisotropic piece in Eq. (7.27) is

$$\mathcal{M}(\mathbf{k};\mathbf{q}) \;=\; \frac{f_{\rm res}}{2}\,\mathcal{F}_{\rm res}(k,q,|\mathbf{k}-\mathbf{q}|)\left[P_2(\hat{\mathbf{k}}\cdot\hat{\chi}^{(0)})+P_2(\hat{\mathbf{q}}\cdot\hat{\chi}^{(0)})+P_2(\widehat{\mathbf{k}-\mathbf{q}}\cdot\hat{\chi}^{(0)})\right], \tag{7.32}$$

  with $f_{\rm res}$ setting the overall non–Gaussian amplitude and $\mathcal{F}_{\rm res}$ a symmetric k–space envelope peaked near the interference band ($k\sim k_\star$).

By construction, $\delta_{\rm IC}$ has the target $P(\mathbf{k}) = |\mathcal{T}|^2P_{\rm iso}(k)$ and reproduces the leading anisotropic bispectrum moment $g_*\propto f_{\rm res}$. Setting $f_{\rm res}=0$ and $k_{\star,\parallel}=k_{\star,\perp}$ restores the isotropic Gaussian limit.

Practical generation. One draws $\delta_{\rm G}$ on a cubic lattice, multiplies by $\mathcal{T}$ in Fourier space, computes the quadratic convolution in Eq. (7.30) via FFTs, and finally rescales to match the desired σ_8 at $z=0$. The director $\hat{\chi}^{(0)}$ is held fixed across the volume (or given a very long–wavelength stochastic modulation if desired), and multiple realizations are built for variance control.

7.3.2 Semi–analytic evolution: anisotropic LPT

In the Zeldovich/2LPT picture, the Eulerian map is

$$\mathbf{x}(\mathbf{q},a) \;=\; \mathbf{q}+D_1(a)\,\mathbf{s}^{(1)}(\mathbf{q})+D_2(a)\,\mathbf{s}^{(2)}(\mathbf{q})+\cdots, \tag{7.33}$$

with $\nabla_q\cdot\mathbf{s}^{(1)}=-\delta_{\rm IC}$ and standard quadratic kernels for $\mathbf{s}^{(2)}$. In our framework the *growth is weakly anisotropic* due to the residual director,

$$D_1(a,\mu) \;=\; D_1^{\rm iso}(a)\left[1+\epsilon_D(a)\,P_2(\mu)\right], \qquad |\epsilon_D|\ll 1, \tag{7.34}$$

where ϵ_D is a calculable functional of the early–time spectral parameters $(c_{\parallel,\perp},\sigma_{\rm diss})$ (Part I). To leading order, one may implement Eq. (7.34) by evolving modes with an effective growth $D_1(a,\mu)$ and using standard 2LPT kernels; the induced corrections to mode–coupling are $\mathcal{O}(\epsilon_D)$ and can be included perturbatively if needed. This semi–analytic evolution preserves the distinctive alignment of ridges and the angular bispectrum modulation while remaining orders–of–magnitude cheaper than an N–body run.

7.3.3 Halo model with director–dependent bias and profiles

On mildly non–linear scales, we generalize the halo model:

$$P(\mathbf{k}) = P_{1h}(\mathbf{k})+P_{2h}(\mathbf{k}), \tag{7.35}$$

$$P_{1h}(\mathbf{k}) = \int dM\,\frac{dn}{dM}\,|\tilde{u}(\mathbf{k}|M,\mu)|^2\Big/\bar{\rho}^2, \tag{7.36}$$

$$P_{2h}(\mathbf{k}) = \left[\int dM\,\frac{dn}{dM}\,b_1(M,\mu)\,\tilde{u}(\mathbf{k}|M,\mu)\Big/\bar{\rho}\right]^2 P_{\rm lin}(\mathbf{k}), \tag{7.37}$$

where $\tilde{u}$ is the Fourier transform of the halo profile and $\mu = \hat{\mathbf{k}}\cdot\hat{\chi}^{(0)}$. Spectral resonance induces:

$$b_1(M,\mu) = b_1^{\rm iso}(M)\left[1+\beta_b(M)P_2(\mu)\right], \qquad c(M,\theta) = c_0(M)\left[1+\beta_c(M)P_2(\cos\theta)\right], \tag{7.38}$$

with θ the angle between halo major axis and $\hat{\chi}^{(0)}$. The *joint* measurement of β_b (large–scale clustering), β_c (stacked lensing shapes), and the bispectrum coefficient g_* provides an over–constrained test: in the resonance picture all three trace back to the same interference band parameters $(k_{\star,\parallel}, k_{\star,\perp}, \Sigma_{\parallel,\perp})$.

7.3.4 N–body emulation and parameter inference

We define a compact parameter set

$$\Theta \equiv \left\{k_{\star,\parallel}, k_{\star,\perp}, \Sigma_\parallel, \Sigma_\perp,\ f_{\rm res},\ \epsilon_D\right\} \tag{7.39}$$

(and standard cosmological parameters). A space–filling design (Latin hypercube) in Θ is used to generate ICs via Eqs. (7.30)–(7.32) and evolve them with N–body or anisotropic 2LPT to $z \in \{2,1,0\}$. From each realization we extract a summary vector

$$\mathbf{S} = \left\{\mathcal{A}_\nabla(R_i),\ \mathcal{V}_\mu(\nu_j;R_i),\ \gamma_{\rm skel}(R_i),\ B(k_a,k_b,k_c;\mu\text{–bins}),\ \text{alignment stats}\right\}. \tag{7.40}$$

A Gaussian–process (or neural) emulator is trained to map $\Theta \mapsto \mathbf{S}$. Likelihood–based inference proceeds with survey summaries $\mathbf{S}_{\rm data}$ and a covariance from isotropic mocks, enabling joint constraints on (Θ, cosmo). Crucially, a *null–restoration test* with $\epsilon_D = f_{\rm res} = 0$ quantifies how much of $\mathbf{S}_{\rm data}$ is already explained by ΛCDM + bias.

7.3.5 Validation metrics and end–to–end diagnostics

Beyond power spectra we track:

- **Genus/percolation shifts:** $\Delta\nu_\chi(R)$ and $\Delta\nu_{\rm perc}(R)$ at fixed $P(k)$.
- **Filamentarity and alignment:** shapefinder filamentarity vs. ν and the alignment distribution $p(\cos\theta)$ of filaments/halo axes relative to $\hat{\chi}^{(0)}$.
- **Angular bispectrum:** the coefficient g_* (Sec. **??**) in squeezed and equilateral bins.
- **Skeleton anisotropy:** $\gamma_{\rm skel}(R)$ (Eq. (7.25)) with error budgets from survey masks and RSD, validated on isotropic mocks.

A pipeline "T–T–Q" test (Theory → Training → Quality) is enforced: (i) theory consistency gates (no superluminal growth; restoration to isotropy when $k_{\star,\parallel} = k_{\star,\perp}$), (ii) emulator cross–validation on held–out simulations, (iii) quality metrics comparing full $\mathbf{S}$ between data and best–fit *and* its isotropic surrogate with matched $P(k)$.

Summary. The recipe above operationalizes the comparison between the spectral–resonance cosmology and standard structure–formation tools. By embedding anisotropic transfer and controlled non–Gaussian phase coupling directly in ICs, and by carrying forward alignment–sensitive diagnostics (skeleton anisotropy, genus/percolation shifts, angular bispectrum), we obtain a suite of discriminants that remain robust under N–body evolution and cannot be faked by isotropic Gaussian initial conditions with the same $P(k)$.

Part IV

Late-Time and Dark Sector

Chapter 8

Late-Time Acceleration from Residual η-Relaxation

8.1 Effective Equation of State $w_{\rm eff}(t) = -1 + \delta(t)$

In the Quantum–Aether framework, dark–energy–like behavior arises naturally from the slow, residual relaxation of the spectral–viscosity field η on cosmological scales. After the early inflationary η–wave has decayed and reheating has completed (Part II), the field typically sits near a shallow minimum of its effective potential $V_\eta(\eta)$ with a small but nonvanishing drift $\dot\eta \neq 0$ driven by the finite spectral friction and the Hubble drag. This section develops a precise *effective–fluid* description, derives the equation of state

$$w_{\rm eff}(t) \;=\; -1 + \delta(t)\,,$$

and provides model–independent mappings to observables ($H(z)$, distances, growth rates), together with theoretical consistency gates that any late–time η–relaxation history must satisfy.

8.1.1 Effective–fluid mapping for the η sector

The late–time Lagrangian density of η (coarse–grained in the aether frame u^μ) is

$$\mathcal{L}_\eta \;=\; \frac{\kappa_\eta(\mathcal{S})}{2}\, g_{\rm eff}^{\mu\nu}\, \nabla_\mu\eta\, \nabla_\nu\eta \;-\; V_\eta(\eta) \;-\; \frac{\Gamma_\eta(\mathcal{S})}{2}\,(u^\mu\nabla_\mu\eta)^2, \tag{8.1}$$

with positive kinetic normalization $\kappa_\eta > 0$ and nonnegative Rayleigh coefficient $\Gamma_\eta \geq 0$ (entropy gate). The conservative stress tensor (excluding the explicitly dissipative term) is

$$T^{(\eta)}_{\mu\nu} \;=\; \kappa_\eta\, \nabla_\mu\eta\, \nabla_\nu\eta \;-\; g^{\rm eff}_{\mu\nu}\Big(\frac{\kappa_\eta}{2}(\nabla\eta)^2 + V_\eta(\eta)\Big)\,, \qquad (\nabla\eta)^2 \equiv g^{\alpha\beta}_{\rm eff}\nabla_\alpha\eta\,\nabla_\beta\eta. \tag{8.2}$$

In a homogeneous background $\eta(t)$ one has

$$\rho_\eta = \frac{\kappa_\eta}{2}\dot\eta^2 + V_\eta(\eta), \qquad p_\eta = \frac{\kappa_\eta}{2}\dot\eta^2 - V_\eta(\eta), \qquad w_\eta \equiv \frac{p_\eta}{\rho_\eta}. \tag{8.3}$$

It is convenient to define the kinetic fraction $K \equiv \kappa_\eta\dot\eta^2/2$ and the *deviation parameter*

$$\delta(t) \;\equiv\; 1 + w_\eta(t) \;=\; \frac{2K}{K+V_\eta} \;=\; \frac{\kappa_\eta\,\dot\eta^2}{V_\eta}\,\Big[1 + \mathcal{O}\Big(\frac{K}{V_\eta}\Big)\Big]. \tag{8.4}$$

Late–time acceleration corresponds to $K \ll V_\eta$, so $\delta(t) \ll 1$ and $w_\eta \simeq -1 + \delta$.

Background dynamics. Variation of (8.1) yields the homogeneous equation of motion

$$\kappa_\eta\,\ddot\eta \;+\; \big(3H\,\kappa_\eta + \dot\kappa_\eta + \Gamma_\eta\big)\,\dot\eta \;+\; V_\eta'(\eta) \;=\; 0, \tag{8.5}$$

with the emergent Hubble rate $H \equiv \frac{1}{3}\nabla_\mu u^\mu$ determined by the effective Einstein equation (Vol. I, Ch. 3). The total expansion obeys

$$H^2(z) \;=\; \frac{1}{3M_{\rm eff}^2(\rho)}\left[\rho_m^0(1+z)^3+\rho_r^0(1+z)^4+\rho_k^0(1+z)^2+\rho_\eta(z)\right], \tag{8.6}$$

where $M_{\rm eff}(\rho)$ is the (slowly varying) effective Planck mass and $\rho_k^0 \propto -\Omega_{k,0}$ if spatial curvature is included.

8.1.2 Analytical control near a shallow minimum

Assume the late–time potential admits a regular expansion around its asymptotic value,

$$V_\eta(\eta) \;=\; V_\Lambda \;+\; \frac{1}{2}\,m_\eta^2\,(\eta-\eta_\infty)^2 \;+\; \mathcal{O}\big((\eta-\eta_\infty)^3\big), \qquad V_\Lambda>0,\; m_\eta^2>0, \tag{8.7}$$

and that $M_{\rm eff}$ and κ_η vary slowly on Hubble time scales. In the overdamped regime

$$3H\,\kappa_\eta+\Gamma_\eta \;\gg\; m_\eta\,, \tag{8.8}$$

Eq. (8.5) reduces to

$$\dot\eta \;\simeq\; -\frac{V_\eta'}{3H\,\kappa_\eta+\Gamma_\eta} \;=\; -\frac{m_\eta^2}{3H\,\kappa_\eta+\Gamma_\eta}\,(\eta-\eta_\infty)\,, \tag{8.9}$$

so that the displacement decays as

$$\eta(t)-\eta_\infty \;\propto\; \exp\left[-\int^t dt'\,\frac{m_\eta^2}{3H(t')\,\kappa_\eta+\Gamma_\eta}\right]. \tag{8.10}$$

Substituting (8.9) into (8.4) yields the leading deviation

$$\delta(t) \;\simeq\; \frac{\kappa_\eta\,m_\eta^4}{\big(3H\,\kappa_\eta+\Gamma_\eta\big)^2}\,\frac{(\eta-\eta_\infty)^2}{V_\Lambda} \;\propto\; \exp\left[-2\int^t dt'\,\frac{m_\eta^2}{3H\,\kappa_\eta+\Gamma_\eta}\right]. \tag{8.11}$$

Thus $\delta(t)$ decays monotonically (up to slow variations of $H,\kappa_\eta,\Gamma_\eta$), with two useful limits:

$$\text{(friction–dominated)}\quad \Gamma_\eta \gg 3H\kappa_\eta: \quad \delta(t) \;\propto\; \exp\big[-2\,(m_\eta^2/\Gamma_\eta)\,t\big]\,, \tag{8.12}$$

$$\text{(Hubble–dominated)}\quad 3H\kappa_\eta \gg \Gamma_\eta: \quad \delta(a) \;\propto\; \exp\left[-\frac{2m_\eta^2}{3\kappa_\eta}\int^a \frac{da'}{a'H(a')^2}\right]. \tag{8.13}$$

In either case, $w_{\rm eff}\to -1$ at late times, with the approach rate set by $(m_\eta^2,\kappa_\eta,\Gamma_\eta)$.

8.1.3 Model–independent mapping to $H(z)$ and distances

Treating the η sector as an effective fluid with $w(z)=-1+\delta(z)$, its energy density obeys

$$\frac{d\rho_\eta}{dz} \;=\; \frac{3}{1+z}\big[1+w(z)\big]\rho_\eta \;=\; \frac{3}{1+z}\,\delta(z)\,\rho_\eta, \qquad \rho_\eta(z) \;=\; \rho_{\eta,0}\,\exp\left[3\int_0^z \frac{\delta(z')}{1+z'}\,dz'\right]. \tag{8.14}$$

Hence the expansion history is

$$E^2(z) \;\equiv\; \frac{H^2(z)}{H_0^2} \;=\; \Omega_{m0}(1+z)^3+\Omega_{r0}(1+z)^4+\Omega_{k0}(1+z)^2+\Omega_{\eta0}\,f_\eta(z), \tag{8.15}$$

with

$$f_\eta(z) \;=\; \exp\left[3\int_0^z \frac{\delta(z')}{1+z'}\,dz'\right], \qquad \Omega_{\eta0}\equiv\frac{\rho_{\eta,0}}{3M_{\rm eff}^2H_0^2}\,. \tag{8.16}$$

Standard distances follow: $D_C(z)=\int_0^z dz'/H(z')$, $D_L=(1+z)D_M$, etc. For data analyses it is convenient to parameterize $\delta(z)$ as a controlled series:

$$\text{(CPL form)}\qquad w(z)=w_0+w_a\frac{z}{1+z} \;\Rightarrow\; \delta(z)=(1+w_0)+w_a\frac{z}{1+z}, \tag{8.17}$$

$$\text{(log form)}\qquad \delta(z)=\delta_0+\delta_1\ln(1+z) \quad \text{(useful for small deviations).} \tag{8.18}$$

Near a shallow minimum (8.7) one finds, to leading order in slow roll,

$$1 + w_0 \;\simeq\; \frac{\kappa_{\eta,0}}{V_{\Lambda,0}} \frac{(V_\eta')_0^2}{\big(3H_0\kappa_{\eta,0} + \Gamma_{\eta,0}\big)^2}, \qquad w_a \;\simeq\; -\frac{2M_{\mathrm{eff},0}^2\,\kappa_{\eta,0}\,V_\eta''}{V_{\Lambda,0}} \left(\frac{3H_0}{3H_0 + \Gamma_{\eta,0}/\kappa_{\eta,0}}\right)^2 \times \mathcal{F}, \quad (8.19)$$

where $\mathcal{F}$ encodes order–unity corrections from $d\kappa_\eta/dt$ and dM_{eff}/dt; the signs indicate the generic trend $w_a < 0$ as the system relaxes toward -1.

8.1.4 Growth of structure and the effective slip

Even if $c_T \simeq 1$ (Sec. 14.3), a varying $\rho_\eta(z)$ modifies the background and thus the linear growth. The growth factor $D(a)$ satisfies

$$D''(a) + \left[\frac{3}{a} + \frac{d\ln H}{d\ln a}\frac{1}{a}\right] D'(a) - \frac{3}{2}\frac{\Omega_{m0}H_0^2}{a^5 H^2(a)}\,\mu_{\mathrm{eff}}(a)\,D(a) = 0, \quad (8.20)$$

where $\mu_{\mathrm{eff}}(a)$ captures any residual, small modification of the Poisson coupling from the structural sector (typically $\mu_{\mathrm{eff}} = 1 + \mathcal{O}(10^{-2})$ if M_{eff} is nearly constant). A useful observable compression is the growth index γ defined by $f \equiv d\ln D/d\ln a \simeq \Omega_m(a)^\gamma$, which for small δ and $\mu_{\mathrm{eff}} - 1$ admits

$$\gamma \;\simeq\; \frac{3(1 - w_{\mathrm{eff}})}{5 - 6w_{\mathrm{eff}}} \;+\; \frac{3}{125}\big(1 - w_{\mathrm{eff}}\big)\big(1 - \Omega_m(a)\big)\,(\mu_{\mathrm{eff}} - 1) \;+\; \cdots . \quad (8.21)$$

Combining $H(z)$, distances, and $f\sigma_8$ thus constrains (δ_0, δ_1) jointly with any allowed drift in M_{eff}.

8.1.5 Consistency gates and bounds

Late–time η–relaxation must pass:

- **(L1) Positivity & stability:** $\kappa_\eta > 0$, $V_\eta'' \geq 0$ at late times; no ghosts in the tensor/scalar sectors; $\sigma_{\mathrm{diss}} \geq 0$ from $\Gamma_\eta \geq 0$.
- **(L2) Near–Λ behavior:** $\delta(t) \ll 1$ for $z \lesssim \mathcal{O}(1)$; practically, $|1 + w_0|$ and $|w_a|$ must lie in the current "few–percent" regime to match distance data.
- **(L3) Gravitational–wave bounds:** any coupling of η to the curvature sector must keep $c_T \simeq 1$ at late times (Sec. 14.3); likewise, $|\dot{M}_{\mathrm{eff}}|/(HM_{\mathrm{eff}}) \ll 1$ to satisfy background and GW luminosity–distance constraints.
- **(L4) Isotropy:** residual alignment with the internal director $\hat{\chi}^{(0)}$ must be volume–averaged to negligible large–angle anisotropy in the Hubble flow and in $w(z, \theta)$.
- **(L5) Restoration:** in the formal limit $\Gamma_\eta \to \infty$ or $m_\eta \to 0$ (with $V_\eta \to V_\Lambda$) the theory reduces continuously to ΛCDM.

Summary. Residual η–relaxation provides a controlled, falsifiable departure from a rigid cosmological constant. The single deviation function $\delta(z)$ encapsulates the late–time dynamics and maps cleanly into $H(z)$, distances, and growth. Its amplitude and evolution are set by three physical knobs—$(m_\eta, \kappa_\eta, \Gamma_\eta)$—subject to tight theoretical gates (positivity, $c_T \simeq 1$, small drift in M_{eff}). This affords a direct bridge between spectral microphysics and low–redshift cosmology.

8.2 Environmental Correlations with ρ and Lensing

Residual relaxation of the spectral–viscosity condensate η at late times need not be perfectly homogeneous. Because the normalization $\kappa_\eta(\mathcal{S})$, the Rayleigh coefficient $\Gamma_\eta(\mathcal{S})$, and even the shallow curvature $V_\eta''(\eta)$ may depend weakly on the *structural state* $\mathcal{S} = \{\rho, \chi^\mu, \dots\}$ of the aether, large–scale fluctuations in the density sector $\delta\rho$ act as sources for small inhomogeneities in η. This induces a *spatially modulated* deviation from $w = -1$,

$$w_{\mathrm{eff}}(\mathbf{x}, a) \;=\; -1 + \delta_0(a) \;+\; \delta_{\mathrm{env}}(\mathbf{x}, a)\,,$$

with a characteristic *scale dependence*. The purpose of this section is to (i) derive the linear response $\delta\eta$ to matter inhomogeneities, (ii) propagate it to $\delta_{\rm env}$ and to weak–lensing observables, and (iii) identify robust, low–redshift tests based on cross–correlations between distance indicators, galaxy density, and lensing maps.

8.2.1 Linear response of η to matter inhomogeneities

Linearizing the late–time equation of motion (8.5) about the homogeneous background $\bar{\eta}(a)$ and working in Newtonian gauge for the effective metric $g_{\rm eff}$, we write

$$\kappa_\eta\,\ddot{\delta\eta} + (3H\kappa_\eta + \dot{\kappa}_\eta + \Gamma_\eta)\,\dot{\delta\eta} - \kappa_\eta\, c_\eta^2(a,\hat{\mathbf{k}}\cdot\hat{\chi}^{(0)})\,\frac{\nabla^2}{a^2}\delta\eta + m_{\eta,\rm eff}^2(a)\,\delta\eta \;=\; \mathcal{S}_\eta[\delta\rho,\Phi,\Psi]\,. \tag{8.22}$$

Here c_η^2 is the (slightly anisotropic) spectral sound speed (Sec. 14.1), $m_{\eta,\rm eff}^2 \equiv V_\eta''(\bar{\eta}) + \mathcal{O}(H\dot{\bar{\eta}})$, and the source term collects the *environmental* dependences,

$$\mathcal{S}_\eta \;\equiv\; -\frac{\partial V_\eta'}{\partial\rho}\,\delta\rho - \frac{\partial\Gamma_\eta}{\partial\rho}\,\dot{\bar{\eta}}\,\dot{\delta\eta} - \frac{\partial\kappa_\eta}{\partial\rho}\,(\ddot{\bar{\eta}} + 3H\dot{\bar{\eta}})\,\delta\rho \;+\; \text{metric–mixing terms}\,, \tag{8.23}$$

all evaluated on the background. In the *quasi–static* sub–Hubble regime ($k \gg aH$) and for slowly varying coefficients, (8.22) reduces in Fourier space to a Yukawa–type screening equation,

$$\left[k^2 + a^2\lambda_\eta^{-2}(a,\hat{\mathbf{k}})\right]\delta\eta(\mathbf{k},a) \;\simeq\; a^2\,\alpha_\eta(a)\,\delta\rho(\mathbf{k},a)\,, \qquad \lambda_\eta^{-2} \;\equiv\; \frac{m_{\eta,\rm eff}^2}{\kappa_\eta c_\eta^2}\,, \tag{8.24}$$

with an *environmental coupling*

$$\alpha_\eta(a) \;\equiv\; \frac{1}{\kappa_\eta c_\eta^2}\left(\frac{\partial V_\eta'}{\partial\rho} + \frac{\partial\kappa_\eta}{\partial\rho}\,\frac{V_\eta'}{\kappa_\eta}\right)\Bigg|_{\bar{\eta},\bar{\rho}}\,. \tag{8.25}$$

Equations (8.24)–(8.25) summarize the *model–independent* response: $\delta\eta$ follows $\delta\rho$ with a scale–dependent kernel characterized by the *screening length* λ_η and amplitude α_η.

8.2.2 From $\delta\eta$ to a modulated equation of state

The local deviation $w_{\rm eff} = -1 + \delta$ inherits an environmental piece through the kinetic and potential contributions,

$$\delta_{\rm env}(\mathbf{k},a) \;\equiv\; \frac{\partial\delta}{\partial\eta}\,\delta\eta(\mathbf{k},a) \;+\; \frac{\partial\delta}{\partial\dot{\eta}}\,\delta\dot{\eta}(\mathbf{k},a) \;\simeq\; \beta_\eta(a)\,\frac{1}{1+k^2\lambda_\eta^2(a)}\,\delta_m(\mathbf{k},a)\,, \tag{8.26}$$

where $\delta_m \equiv \delta\rho/\bar{\rho}_m$, and the *response amplitude*

$$\beta_\eta(a) \;\equiv\; \frac{\bar{\rho}_m(a)}{\bar{V}_\eta(a)}\left[\kappa_\eta\,\dot{\bar{\eta}}\,\frac{\partial\delta}{\partial\dot{\eta}}\,\frac{\alpha_\eta}{3H\kappa_\eta + \Gamma_\eta} \;-\; \frac{\partial\delta}{\partial\eta}\,\alpha_\eta\right]_{\bar{\eta},\bar{\rho}}\,, \tag{8.27}$$

encodes the microphysical dependence (sign and magnitude) of the modulation. Two robust qualitative consequences follow:

1. *Scale dependence:* on scales $k\lambda_\eta \gg 1$ the modulation is screened; on very large scales ($k\lambda_\eta \ll 1$) it approaches a constant bias $\delta_{\rm env} \to \beta_\eta\,\delta_m$.
2. *Universality:* if α_η depends only on the *total* energy density (through ρ), then the environmental response is universal and compatible with the weak equivalence principle at linear order.

8.2.3 Cross–correlations with weak lensing and galaxy density

Weak lensing convergence $\kappa(\hat{\mathbf{n}},z_s)$ probes the line–of–sight integral of the lensing potential $\Phi_{\rm L} \equiv (\Phi+\Psi)/2$. In our framework, two channels imprint the η modulation:

1. **Background channel:** variations in $\rho_\eta(z)$ (through $\delta_0(a)$) alter $H(z)$ and the lensing kernels $W_\kappa(z_s; z)$ (Sec. 14.1).
2. **Environmental channel:** a small, scale–dependent change of the *effective* Poisson coupling $\mu_{\rm eff}(k,a)$ can arise if $M_{\rm eff}(\rho)$ drifts with ρ or if the structural sector contributes a tiny slip. Linearizing,

$$k^2\Phi_{\rm L} \;=\; 4\pi G\, a^2\, \mu_{\rm eff}(k,a)\, \bar\rho_m\, \Delta_m, \qquad \mu_{\rm eff}(k,a) \;=\; 1+\nu_\eta(a)\,\frac{1}{1+k^2\lambda_G^2(a)}\,, \tag{8.28}$$

where λ_G is a (generically larger) *gravitational screening* length and $|\nu_\eta| \ll 1$ by current bounds.

The galaxy–lensing angular cross–power between a lens bin i and a source bin j then reads

$$C_\ell^{g_i\kappa_j} \;=\; \int_0^\infty \frac{dz}{H(z)}\,\frac{W_{g_i}(z)\,W_{\kappa_j}(z)}{\chi^2(z)}\,\mu_{\rm eff}\!\left(\frac{\ell}{\chi},z\right)\,P_{mm}\!\left(k=\frac{\ell}{\chi},z\right), \tag{8.29}$$

where W_{g_i} includes the galaxy bias $b_g(z)$ and W_{κ_j} is the standard lensing kernel. A joint fit to $\{C_\ell^{gg}, C_\ell^{g\kappa}, C_\ell^{\kappa\kappa}\}$ constrains (ν_η, λ_G).

Hubble–diagram residuals and lensing maps. A clean probe of the *environmental w–modulation* is the cross–correlation between standardized–candle residuals (e.g. SN Ia) and lensing or density tracers. For a sample in bin b at effective redshift z_b, the magnitude residual Δm sourced by $\delta_{\rm env}$ is

$$\Delta m(\hat{\mathbf{n}}) \;\simeq\; \frac{5}{\ln 10}\,\frac{1}{2}\int_0^{z_b} dz\,\frac{\partial \ln H^{-2}}{\partial w}\,\delta_{\rm env}(\chi(z)\hat{\mathbf{n}},z)\,, \tag{8.30}$$

with $\partial \ln H^{-2}/\partial w$ evaluated on the background. Using (8.26), the SN–convergence cross–spectrum becomes

$$C_\ell^{m\kappa}(z_b) \;\propto\; \int_0^{z_b} \frac{dz}{H(z)}\,\frac{\partial \ln H^{-2}}{\partial w}\,\frac{\beta_\eta(z)}{1+\lambda_\eta^2(z)\,(\ell/\chi)^2}\,\frac{W_\kappa(z;z_b)}{\chi^2(z)}\,P_{mm}\!\left(\frac{\ell}{\chi},z\right). \tag{8.31}$$

A detection of a *scale–dependent* $C_\ell^{m\kappa}$ following the Yukawa kernel would be a smoking gun for environmental η–modulation.

8.2.4 Forecast parameters and priors

For data–driven analyses we recommend the minimal parameterization

$$\boxed{\{\delta_0,\ \delta_1\}\ \text{for background}\quad;\quad \{\beta_{\eta,0},\ \lambda_{\eta,0}\}\ \text{for environmental modulation}\quad;\quad \{\nu_{\eta,0},\ \lambda_{G,0}\}\ \text{for weak–gravity slip}} \tag{8.32}$$

with mild power–law or CPL–like redshift drifts if required by the data. Consistency with Sec. 14.1 suggests Gaussian priors $|\delta_0|, |\delta_1| \lesssim 0.05$, $|\nu_{\eta,0}| \lesssim 10^{-2}$, and broad, positivity priors $\beta_{\eta,0} \in \mathbb{R}$, $\lambda_{\eta,0} > 0$.

8.2.5 Consistency gates for environmental effects

- **(E1) Smallness:** $|\beta_\eta|\,\sigma_8 \ll 1$ and $|\nu_\eta| \ll 1$ to satisfy isotropy of the Hubble flow and current lensing constraints.
- **(E2) Universality:** α_η must couple to *total* density at linear order to preserve weak equivalence.
- **(E3) Screening hierarchy:** $\lambda_G \gtrsim \lambda_\eta$ so that gravity–sector deviations are at most comparable (and typically subdominant) to the w–modulation kernel.
- **(E4) Restoration:** in the limit $(\beta_\eta, \nu_\eta) \to 0$ the standard ΛCDM relations are recovered for all cross–spectra.

Summary. Late–time residual η–relaxation predicts *environment–dependent* departures from $w = -1$ with a calculable Yukawa screening scale λ_η and amplitude β_η. These induce distinctive, scale–dependent cross–correlations between standardized–candle residuals, galaxy density, and weak–lensing maps, while keeping gravity–sector deviations within current bounds. Joint tomographic analyses of $\{H(z), D(z), f\sigma_8(z), C_\ell^{g\kappa}, C_\ell^{m\kappa}\}$ can therefore separate background evolution (δ_0, δ_1) from environmental response $(\beta_\eta, \lambda_\eta)$, providing a sharp test of the spectral aether cosmology at late times.

8.3 Constraints from GW Propagation and Multi–Messenger Bounds

Tensor perturbations of the emergent metric $g_{\mu\nu}^{\rm eff}$ obey, to leading order,

$$\ddot{h}_{ij} + \big(3H + \alpha_M^{\rm spec} H + \gamma_T\big)\,\dot{h}_{ij} \;+\; c_T^2(\theta)\,\frac{k^2}{a^2}\,h_{ij} \;+\; m_T^2\,h_{ij} \;=\; \frac{2}{M_{\rm eff}^2(\rho)}\,\Pi_{ij}^{(\rm TT)}\,, \tag{8.33}$$

in the aether frame (u^μ), where: (i) $c_T(\theta) = 1+\delta c_T(\theta)$ is the GW phase speed with a (small) anisotropy inherited from $\hat{\chi}^{(0)}$; (ii) $\alpha_M^{\rm spec} \equiv d\ln M_{\rm eff}^2/d\ln a$ encodes the running of the emergent Planck mass;[1] (iii) $\gamma_T \geq 0$ is a Rayleigh–type damping induced by residual spectral couplings; (iv) m_T is an effective graviton mass (set to zero in the minimal model); and $\Pi_{ij}^{(\rm TT)}$ denotes transverse–traceless sources. Equation (8.33) reduces to GR when $\{\delta c_T, \alpha_M^{\rm spec}, \gamma_T, m_T\} \to 0$.

8.3.1 Phase speed constraints (local and cosmological)

A line–of–sight integration of the group–velocity difference between GWs and photons gives the GW–EM arrival–time offset for a source at redshift z_s:

$$\Delta t_{\rm GW-EM}(z_s, \hat{\mathbf{n}}) = \int_0^{z_s} \frac{dz}{(1+z)\,H(z)} \left[\frac{1}{c_T\big(z, \theta(\hat{\mathbf{n}}, z)\big)} - 1\right]. \tag{8.34}$$

Multi–messenger events with $\Delta t_{\rm GW-EM} \approx 0$ imply

$$\big|\,\delta c_T(z \lesssim \text{few} \times 10^{-2})\,\big| \;\ll\; 1 \qquad \text{(luminal within very tight bounds).} \tag{8.35}$$

In addition, the absence of gravitational Cherenkov losses for ultra–high–energy cosmic rays enforces $c_T \geq 1 - \varepsilon_{\rm Ch}$ with $\varepsilon_{\rm Ch} \ll 1$.[2] The spectral model must therefore realize

$$\delta c_T(\theta) \;=\; \underbrace{\Xi_1 \frac{(\partial\eta)^2}{M_{\rm eff}^2}}_{\text{viscosity back–reaction}} \;+\; \underbrace{\Xi_2\,(\hat{\mathbf{k}}\cdot\hat{\chi}^{(0)})^2}_{\text{director anisotropy}} \;+\; \cdots \quad\Rightarrow\quad \big|\delta c_T\big| \text{ and its quadrupole } \ll 1 \text{ today.} \tag{8.36}$$

8.3.2 Amplitude damping and the GW luminosity distance

Equation (8.33) implies a modified friction. Writing the WKB solution $h_{ij} \propto \exp\big[-\frac{1}{2}\int(3 + \alpha_M^{\rm spec} + \gamma_T/H)\,d\ln a\big]$ yields a GW luminosity distance

$$d_L^{\rm GW}(z) \;=\; d_L^{\rm EM}(z)\,\exp\left[\frac{1}{2}\int_0^z \frac{\alpha_M^{\rm spec}(z') + \gamma_T(z')/H(z')}{1+z'}\,dz'\right]. \tag{8.37}$$

Standard–siren measurements constrain the integrated combination in the exponent. In the spectral aether, a slow drift of $M_{\rm eff}(\rho)$ produces $\alpha_M^{\rm spec} \neq 0$, whereas *residual* η–mixing yields $\gamma_T > 0$. Consistency with the late–time sector (Sec. 14.1) requires

$$\left|\int_0^z \frac{\alpha_M^{\rm spec} + \gamma_T/H}{1+z'}\,dz'\right| \;\ll\; 1 \quad \text{for } z \lesssim \mathcal{O}(1), \tag{8.38}$$

ensuring $d_L^{\rm GW} \simeq d_L^{\rm EM}$ within current sensitivity.

[1] In our spectral language, $\alpha_M^{\rm spec}$ combines the slow drift of $M_{\rm eff}(\rho)$ with any mixing terms that renormalize the tensor kinetic block.

[2] Empirically, both bounds are extremely stringent in the local universe; we keep them symbolic here to emphasize their role as *gates* on the spectral curvature sector.

8.3.3 Dispersion, birefringence, and polarization mixing

If higher–derivative spectral curvature terms induce dispersion,

$$\omega^2 \;=\; c_T^2 k^2 \;+\; \varsigma_3 \frac{k^3}{M_\star} \;+\; \varsigma_4 \frac{k^4}{M_\star^2} \;+\; m_T^2 \,, \tag{8.39}$$

then broadband GW observations bound $\varsigma_{3,4}$ and m_T.[3] The director field can also source a tiny *birefringence*,

$$c_T^{(\pm)}(\theta) \;=\; 1 \pm \Delta_T \,\sin^2\!\theta \;+\; \cdots \,, \qquad \Delta_T \propto \tilde{\Xi}\, \frac{(\partial\eta)\cdot\hat{\chi}^{(0)}}{M_{\rm eff}} \,, \tag{8.40}$$

rotating polarization and inducing phase shifts between circular modes. Null searches for polarization rotation in compact–binary signals imply $|\Delta_T| \ll 1$ across the LIGO/Virgo/KAGRA band, thereby bounding $\tilde{\Xi}$.

8.3.4 Parameter mapping and priors (spectral curvature sector)

For data confrontation we recommend the minimal late–time parameter set

$$\boxed{\delta c_T(z,\theta) = \delta c_{T,0}\, Y_{00} + \delta c_{T,2}\, Y_{20}(\theta),\;\; \alpha_M^{\rm spec}(z) = \alpha_{M,0} + \alpha_{M,1}\, z/(1+z),\;\; \gamma_T(z) = \gamma_{T,0}\, E(z),\;\; m_T = 0} \tag{8.41}$$

with $E(z) \equiv H(z)/H_0$. The spectral microphysics relates these to Lagrangian couplings as

$$\delta c_{T,0} \sim \Xi_1 \frac{\dot{\bar{\eta}}^{\,2}}{M_{\rm eff}^2 H_0^2} + \Xi_2\, \langle \hat{k}\cdot\hat{\chi}^{(0)}\rangle^2, \quad \alpha_{M,0} \sim \frac{d\ln M_{\rm eff}^2}{d\ln\rho}\, \frac{d\ln\rho}{d\ln a}, \quad \gamma_{T,0} \sim \mathcal{O}(g_{\eta\rm curv}^2)\,, \tag{8.42}$$

all naturally small in the residual–relaxation regime. Priors consistent with multi–messenger null tests are:

$$|\delta c_{T,0}| \ll 1, \qquad |\delta c_{T,2}| \ll 1, \qquad \left| \int_0^1 \frac{\alpha_M^{\rm spec} + \gamma_T/H}{1+z}\, dz \right| \ll 1, \qquad m_T = 0 \text{ (baseline)}. \tag{8.43}$$

8.3.5 Consistency gates (GW sector)

- **(GWT–1) Luminality gate:** $\big|\delta c_T(z \lesssim 0.01)\big| \ll 1$ and no superluminal–subluminal pathologies along observed lines of sight, cf. (8.35).
- **(GWT–2) Damping gate:** the integral in (8.37) is ≈ 0 over $z \lesssim 1$ within current siren uncertainties.
- **(GWT–3) No extra polarizations:** tensor sector remains purely TT at linear order; any scalar–tensor leakage is <percent–level in amplitude.
- **(GWT–4) Weak dispersion/birefringence:** $\varsigma_{3,4} \approx 0$, $m_T = 0$, and $|\Delta_T| \ll 1$ across the detection band.
- **(GWT–5) Restoration:** in the limit of vanishing residual relaxation ($\dot{\bar{\eta}} \to 0$) all parameters return to their GR values exactly.

Summary. The spectral aether predicts a tensor propagation law (8.33) in which *any* deviation from GR is tightly tied to residual late–time relaxation of the viscosity field and to tiny state–dependences of the emergent Planck mass. Multi–messenger timing, standard–siren amplitudes, and broadband GW waveforms thus constrain (and in practice force to be very small) the combinations $\{\delta c_T, \alpha_M^{\rm spec}, \gamma_T, \Delta_T\}$, providing sharp gates on the curvature sector couplings of the model without spoiling its cosmographic successes.

[3] For the minimal spectral Palatini sector we set $m_T = 0$ and $\varsigma_{3,4} = 0$ at tree level; radiative corrections are suppressed by the *curvature gates* of Vol. I.

Chapter 9

Unifying Dark Matter and Dark Energy

9.1 Dark Matter as Stable ρ-Structures

Thesis. In the Quantum–Aether framework, "dark matter" is not a new particulate species appended to spacetime, but a family of *stable, non-luminous, cold* configurations of the density field ρ (possibly braided with the director χ^μ). These configurations are localized (or quasi-localized on halo scales), persist on cosmological times, and interact predominantly through the emergent metric g_{eff} and weak spectral forces. Their bulk properties (mass, core size, self-interaction) and halo-scale phenomenology (cores vs. cusps, substructure survival) are computable in terms of the *structural* couplings already present in the unified action.

9.1.1 Effective structural sector and existence of localized states

At scales well below the inflationary epoch and for slowly varying $M_{\text{eff}}(\rho)$, the structural density sector may be coarse–grained to

$$\mathcal{L}_\rho \;=\; \frac{Z_\rho(\mathcal{S})}{2}\, g_{\text{eff}}^{\mu\nu}\, \nabla_\mu\rho\, \nabla_\nu\rho \;-\; U(\rho) \;-\; \frac{\lambda_4(\mathcal{S})}{4}\, \big(g_{\text{eff}}^{\mu\nu}\, \nabla_\mu\rho\, \nabla_\nu\rho\big)^2 \;+\; \mathcal{L}_{\text{mix}}[\rho,\chi^\mu]\,, \tag{9.1}$$

where $Z_\rho > 0$, $\lambda_4 \geq 0$, and $\mathcal{L}_{\text{mix}}$ collects symmetry-allowed couplings to the internal director and phase (e.g. $(\nabla\rho\cdot\chi)^2$ terms, Faddeev–Skyrme–like quartics induced after integrating out fast modes; cf. Vol. I, Ch. 4). The potential $U(\rho)$ admits a homogeneous background ρ_0 and supports *compressional stiffness* $U''(\rho_0) > 0$.

Derrick-type virial balance. Let E_2 and E_4 denote, respectively, the contributions from the quadratic and quartic gradient terms in a static configuration $\rho(\mathbf{x})$:

$$E_2 \equiv \int d^3x\, \frac{Z_\rho}{2}\, (\nabla\rho)^2, \qquad E_4 \equiv \int d^3x\, \frac{\lambda_4}{4}\, \big[(\nabla\rho)^2\big]^2, \qquad E_U \equiv \int d^3x\, U(\rho) - U(\rho_0). \tag{9.2}$$

Under the scaling $\rho(\mathbf{x}) \mapsto \rho_\lambda(\mathbf{x}) = \rho(\lambda\mathbf{x})$, one finds

$$E(\lambda) = \lambda\, E_2 + \lambda^{-1}\, E_U + \lambda^{-1}\, E_4\,, \quad \frac{dE}{d\lambda}\Big|_{\lambda=1} = E_2 - E_U - E_4 = 0. \tag{9.3}$$

Thus, for localized finite–energy configurations ($E_U > 0$), the stabilizing balance is $E_2 \simeq E_U + E_4$, showing explicitly why *quartic* structural stiffness (or topological mixing in $\mathcal{L}_{\text{mix}}$) is required in 3D to evade collapse (Derrick's theorem). This is the same mechanism sustaining the Faddeev–Skyrme/Hopfion family in Vol. I, Ch. 4.

Characteristic core scale. Defining a compressional mass scale $m_\rho^2 \equiv U''(\rho_0)$ and a dimensionless stiffness $\hat{\lambda}_4 \equiv \lambda_4 Z_\rho^{-2} m_\rho^2$, stationary variational estimates yield a characteristic core radius

$$\ell_\star \sim \frac{\sqrt{\lambda_4}}{Z_\rho}\frac{1}{m_\rho} = \frac{\sqrt{\hat{\lambda}_4}}{m_\rho}, \qquad M_{\rm sol} \sim \rho_0\,\ell_\star^3 + \mathcal{T}\,\ell_\star + \cdots, \tag{9.4}$$

where $\mathcal{T}$ encodes possible string–like tension pieces when ρ couples to χ^μ in ring/tube geometries (cf. Hopfions).

9.1.2 Spherical halo solitons: hydrostatic treatment and profiles

At galactic scales, the coarse–grained stress tensor of the ρ–medium behaves as an anisotropic fluid with an *effective equation of state* determined by the gradient energy:

$$p_\rho \simeq \frac{Z_\rho}{6}\langle(\nabla\rho)^2\rangle + \frac{\lambda_4}{12}\langle(\nabla\rho)^4\rangle \quad\Longrightarrow\quad p_\rho \equiv p_\rho(\rho;\, Z_\rho, \lambda_4, \ldots). \tag{9.5}$$

Assuming approximate spherical symmetry and slow emergent expansion on halo times, hydrostatic balance in the $g_{\rm eff}$–Newtonian limit is

$$\frac{dp_\rho}{dr} = -\frac{G_{\rm eff}M(r)}{r^2}\rho(r), \qquad M(r) = 4\pi\int_0^r \rho(r')\,r'^2 dr', \qquad G_{\rm eff} \equiv \left(8\pi M_{\rm eff}^2(\rho_0)\right)^{-1}. \tag{9.6}$$

For a broad class of gradient–dominated closures, one obtains *cored* solutions. Two useful analytic representatives are:

$$\text{(i) Soliton–core:} \qquad \rho(r) = \frac{\rho_c}{\left(1+\frac{r^2}{r_c^2}\right)^p}, \qquad p \in \left(\tfrac{3}{2}, 2\right), \tag{9.7}$$

$$\text{(ii) Burkert–like:} \qquad \rho(r) = \frac{\rho_0\, r_0^3}{(r+r_0)\,(r^2+r_0^2)}, \tag{9.8}$$

with $r_c \sim \ell_\star$ given by (9.4). The circular velocity $v_c^2(r) = G_{\rm eff}M(r)/r$ computed from (9.7) or (9.8) exhibits a *flat* rise to a maximum and then a gentle decline, matching the core–dominated rotation curves of low–surface–brightness dwarfs while remaining compatible with NFW–like envelopes once baryonic compression and late–infall are included.

Core scaling. Within the soliton–core family, $M_{\rm core} \sim \rho_c r_c^3$ and the structural virial balance implies an approximate scaling

$$\rho_c\, r_c \sim \frac{Z_\rho}{G_{\rm eff}}\frac{1}{\Upsilon(\hat{\lambda}_4, p)}, \tag{9.9}$$

where Υ is a weakly varying dimensionless function fixed by the closure (numerically calibrated in Sec. 9.3). Equation (9.9) provides a concrete route to relate core–cusp phenomenology to spectral–structural couplings (Z_ρ, λ_4) and $M_{\rm eff}$.

9.1.3 Self–interactions and velocity dependence

Two mechanisms generate effective soliton–soliton interactions:

1. **Contact/overlap repulsion** from the quartic gradient energy, leading to a geometric cross section

$$\sigma_{\rm geo} \sim \pi\,\ell_\star^2 \propto \frac{\hat{\lambda}_4}{m_\rho^2}, \qquad \frac{\sigma_{\rm geo}}{M_{\rm sol}} \sim \frac{\pi}{\rho_0\,\ell_\star} \propto \frac{m_\rho}{\rho_0\sqrt{\hat{\lambda}_4}}. \tag{9.10}$$

2. **Mediated exchange** via light spectral modes (in χ^μ or phase–gauge channels), effectively producing Yukawa–type interactions

$$V_{\rm ex}(R) \simeq \pm\frac{\alpha_{\rm spec}}{R}e^{-m_{\rm spec}R}\,\mathcal{F}(\hat{\chi}_1, \hat{\chi}_2) + \frac{\kappa_{\rm so}}{R^3}\mathbf{S}_1\cdot\mathbf{L} + \cdots, \tag{9.11}$$

with mild orientation dependence $\mathcal{F}$ and spin–orbit–like pieces from Berry curvature (cf. Vol. I, Ch. 6–7). The corresponding momentum–transfer cross section exhibits *velocity dependence*:

$$\sigma_T(v) \sim \sigma_0 \frac{\ln\big(1+\xi^2\big) - \frac{\xi^2}{1+\xi^2}}{\xi^2}, \qquad \xi \equiv \frac{m_{\rm spec} v}{\mu\, \alpha_{\rm spec}}, \tag{9.12}$$

with reduced mass μ of the soliton pair. This realizes the phenomenologically desired trend: larger σ_T/m at dwarf velocities ($v \sim$10–100 km/s) and smaller at cluster velocities ($v \sim$1000–3000 km/s).

Benchmark windows. To orient parameter scans, we will use community benchmark corridors for the momentum–transfer cross section per unit mass,

$$\frac{\sigma_T}{M_{\rm sol}} \sim 0.1\text{–}10\ \mathrm{cm}^2\,\mathrm{g}^{-1} \quad \text{(dwarf/LSB scales)}, \qquad \frac{\sigma_T}{M_{\rm sol}} \lesssim 0.1\ \mathrm{cm}^2\,\mathrm{g}^{-1} \quad \text{(cluster scales)}, \tag{9.13}$$

interpreted here as *targets* to be matched by $(\hat{\lambda}_4, \alpha_{\rm spec}, m_{\rm spec};\ \rho_0, m_\rho)$ via Eqs. (9.10)–(9.12), while respecting the cosmological gates of Vol. I (causality, positivity, restoration).

9.1.4 Substructure, survivability, and small–scale cutoffs

Gradient pressure and mediated repulsion introduce a characteristic suppression scale for compact subhalos. A simple estimate follows from equating structural support to tidal stress at pericenter R_p in a host potential with circular speed V_h:

$$\frac{p_\rho}{r_s} \sim \frac{\rho_s V_h^2}{R_p} \quad \Rightarrow \quad M_{\rm min} \sim \rho_s r_s^3 \propto \left(\frac{Z_\rho}{G_{\rm eff}}\right)^{3/2} \mathcal{G}(\hat{\lambda}_4, \ldots), \tag{9.14}$$

where (ρ_s, r_s) are the subhalo core density/size and $\mathcal{G}$ a slowly varying function of closure parameters. Velocity–dependent scattering further evaporates the smallest subhalos in hot environments while leaving field dwarfs intact—an observationally testable dichotomy.

9.1.5 Observational gates at halo and cluster scales

We distill *falsifiable* requirements for ρ–structures to play the role of dark matter:

- **(D1) Stability:** Static/slowly evolving solutions obey the virial identity (9.3) with $E_2, E_4, E_U > 0$; linear spectra free of tachyons/ghosts (Vol. I gates).
- **(D2) Cores with scalable radii:** Existence of cored profiles (9.7) or (9.8) with r_c tracking (9.4); compatibility with dwarf galaxy rotation curves and LSB systems.
- **(D3) Velocity dependence:** A $\sigma_T(v)$ consistent with (9.13) across dwarf / Milky Way / cluster regimes, achievable via the admixture of (9.10) and (9.11).
- **(D4) Lensing morphology:** Projected mass distributions produce lensing maps with central *plateaus* (for cored halos) yet steep enough outer slopes when embedded in evolving large–scale structure (cf. Chap. 7).
- **(D5) Substructure:** A controlled $M_{\rm min}$ (Eq. (9.14)) that neither over–suppresses satellites ("too few") nor leaves NFW–like cusps intact ("too cuspy").
- **(D6) Restoration:** In the post–inflationary, weak–field limit, the emergent metric obeys $c_T \to 1$ and standard GR lensing relations; any residual spectral forces are short–ranged or weak enough to evade solar–system bounds.

Summary. Stable ρ–structures sustained by structural stiffness and (where present) topological mixing provide a natural, calculable dark–matter sector. Core sizes and self–interactions are not arbitrary: they trace back to $(Z_\rho, \lambda_4, m_\rho)$ and, when needed, to light spectral mediators $(\alpha_{\rm spec}, m_{\rm spec})$. The program for the remainder of this chapter is to (i) calibrate these parameters with halo and cluster data, (ii) propagate their implications to lensing and substructure, and (iii) joint–fit them with the late–time η–relaxation sector of Sec. 9.2 to realize a unified dark sector.

9.2 Dark Energy as Ultra–long η Relaxation

Thesis. Late–time acceleration does not require a rigid cosmological constant. In the Quantum–Aether framework it arises from *ultra–long relaxation* of the spectral–viscosity field η on cosmological scales, after the inflationary condensate has decohered. The residual dynamics acts as an *effective fluid* with $w_{\rm eff}(t) \gtrsim -1$, weak time dependence, and negligible anisotropic stress, while respecting restoration gates ($c_T \to 1$ and GR–like lensing in the low–curvature limit).

9.2.1 Effective fluid description and background dynamics

At late times the coarse–grained η sector can be modeled by

$$\mathcal{L}_\eta^{(\rm late)} \;=\; \frac{\kappa_\eta(\mathcal{S})}{2}\, g_{\rm eff}^{\mu\nu}\nabla_\mu\eta\,\nabla_\nu\eta \;-\; V_\eta(\eta) \;-\; \frac{\Gamma_\eta(\mathcal{S})}{2}\,(u^\mu\nabla_\mu\eta)^2, \qquad \kappa_\eta>0,\ \Gamma_\eta\ge 0, \tag{9.15}$$

with state–dependent but slowly varying $M_{\rm eff}(\rho)$ (cf. Vol. I, Ch. 2). In the homogeneous limit,

$$\rho_\eta = \tfrac{1}{2}\kappa_\eta\dot\eta^2 + V_\eta(\eta), \qquad p_\eta = \tfrac{1}{2}\kappa_\eta\dot\eta^2 - V_\eta(\eta), \qquad w_\eta \equiv \frac{p_\eta}{\rho_\eta}. \tag{9.16}$$

The emergent Friedmann–Raychaudhuri system reads

$$H^2 = \frac{1}{3M_{\rm eff}^2(\rho)}\big(\rho_m+\rho_r+\rho_\eta\big), \qquad \dot H = -\frac{1}{2M_{\rm eff}^2(\rho)}\big(\rho_m+\tfrac{4}{3}\rho_r+\rho_\eta+p_\eta\big), \tag{9.17}$$

with matter/radiation separately conserved to an excellent approximation after reheating (Chap. 6).

Variation of (9.15) yields the late–time equation of motion

$$\kappa_\eta\ddot\eta + \big(3H\kappa_\eta + \dot\kappa_\eta + \Gamma_\eta\big)\dot\eta + V_\eta'(\eta) = 0. \tag{9.18}$$

When potential domination holds, $V_\eta \gg \frac{1}{2}\kappa_\eta\dot\eta^2$, one has $w_\eta \simeq -1+\delta$ with

$$\delta(t) \;\equiv\; 1+w_\eta \;\simeq\; \frac{\kappa_\eta\dot\eta^2}{V_\eta} \;=\; \frac{1}{V_\eta}\left(\frac{V_\eta'}{3H\kappa_\eta+\Gamma_\eta}\right)^2, \qquad 0\le\delta\ll 1 \ \text{ for ultra–slow roll.} \tag{9.19}$$

Parametric forms and mapping to data. Two representative model classes for the residual potential are:

$$\text{(Plateau drift)} \quad V_\eta(\eta) = V_\Lambda\Big[1+\varepsilon\,\mathcal{U}(\eta/\eta_\star)\Big],\ |\varepsilon|\ll 1, \tag{9.20}$$

$$\text{(Runaway with brake)} \quad V_\eta(\eta) = V_\Lambda\Big(1+e^{-\eta/\eta_\star}\Big),\ \ \Gamma_\eta \simeq \Gamma_0+\Gamma_1 e^{-\eta/\eta_\star}, \tag{9.21}$$

where V_Λ sets the observed acceleration scale. Around $z\lesssim 2$, the background expansion can be summarized by a CPL–type fit

$$w_\eta(a) = w_0 + w_a(1-a), \qquad w_0\simeq -1+\delta_0, \quad w_a \simeq -\left.\frac{dw_\eta}{d\ln a}\right|_{a=1}, \tag{9.22}$$

with (w_0, w_a) expressed in terms of $(\kappa_\eta, \Gamma_\eta, V_\eta, M_{\rm eff})$ via (9.19). Current datasets require $|w_0+1|\lesssim 0.05$ and $|w_a|\lesssim 0.3$; these are straightforward priors on the spectral parameters.

9.2.2 Linear perturbations and restoration to GR lensing

In Newtonian gauge, $ds^2 = -(1+2\Psi)dt^2 + a^2(1-2\Phi)d\mathbf{x}^2$ with the *effective* metric $g_{\rm eff}$, the late–time η fluctuation obeys, on subhorizon scales,

$$\ddot{\delta\eta} + \big(3H+\Gamma_\eta/\kappa_\eta\big)\dot{\delta\eta} + c_\eta^2\frac{k^2}{a^2}\delta\eta + m_{\rm eff}^2\,\delta\eta \simeq 2V_\eta'\,\Psi + \big(\dot\eta\,\dot\Psi + \cdots\big), \tag{9.23}$$

with $c_\eta^2 \equiv Z_{\eta,\rm tr}/Z_{\eta,\rm time}$ the (isotropized) sound speed inherited from the principal symbol (Vol. I, Ch. 2) and $m_{\rm eff}^2 = V_\eta'' + \mathcal{O}(H\dot\eta)$. The *restoration gate* requires

$$c_\eta^2 \simeq 1, \qquad \big|\Phi-\Psi\big| \ll |\Phi|, \qquad c_T^2\simeq 1, \tag{9.24}$$

ensuring GR–like lensing and GW propagation at late times. Residual departures can be captured by the (μ, Σ) parametrization,

$$k^2\Psi = -4\pi G_{\rm eff} a^2 \mu(a,k)\rho_m \Delta_m, \qquad k^2(\Phi+\Psi) = -8\pi G_{\rm eff} a^2 \Sigma(a,k)\rho_m \Delta_m, \tag{9.25}$$

with $\mu = \Sigma = 1 + \mathcal{O}(\delta)$ across the linear regime in viable models.

9.2.3 Distinguishability from ΛCDM

Although $w_\eta \simeq -1$, ultra–long η relaxation leaves correlated, falsifiable imprints:

1. **Background drift.** A mild, monotonic $\delta(a) = 1 + w_\eta(a) > 0$ produces $H(z)$ curves distinguishable at the percent level for $z \lesssim 1.5$. In CPL language, models (9.20)–(9.21) predict $w_a \gtrsim 0$ for $\Gamma_\eta \gtrsim 0$ (effective friction), a target for SNe Ia+BAO+cosmic–chronometer combinations.

2. **Growth index.** Linear growth $D(a)$ obeys

$$D'' + \left(\frac{3}{a} + \frac{H'}{H}\right) D' - \frac{3}{2}\frac{\Omega_m(a)}{a^2}\mu(a,k) D = 0, \tag{9.26}$$

   with $'$ denoting d/da. For $\mu = 1 + \mathcal{O}(\delta)$ one may write the growth rate $f \equiv d\ln D/d\ln a \simeq \Omega_m^\gamma$ with

$$\gamma \;\simeq\; 0.55 + 0.05\,[1 + w_0 + 0.5\,w_a], \tag{9.27}$$

   predicting a *coherent* offset in $f\sigma_8$ relative to ΛCDM tied to (w_0, w_a).

3. **ISW and lensing cross–correlations.** A slow decay of $\Phi + \Psi$ at late times enhances the ISW–LSS cross signal by $\mathcal{O}(\delta)$, while $\Sigma(a,k)$ close to unity preserves standard weak–lensing kernels—yielding a distinctive ratio of ISW to lensing amplitudes.

4. **Cosmic–clock bounds on $M_{\rm eff}$.** If $M_{\rm eff}(\rho)$ drifts, then $\dot{G}/G = -2\dot{M}_{\rm eff}/M_{\rm eff}$, constrained by pulsar timing and lunar laser ranging to $|\dot{G}/G| \lesssim 10^{-12}\,{\rm yr}^{-1}$. We implement this as a prior on the allowed spectral drift $\dot{M}_{\rm eff}$, keeping $G_{\rm eff}$ effectively constant over late times.

9.2.4 Consistency gates for the late–time sector

We summarize viability conditions to be enforced in joint fits (Sec. 9.3):

- **(L1) Positivity and stability:** $\kappa_\eta > 0$, $V_\eta \geq 0$, $m_{\rm eff}^2 \geq -\epsilon H^2$ (no tachyonic runaways), and $c_\eta^2 \in (0, 1]$.
- **(L2) Restoration to GR lensing:** $|\Phi - \Psi| \ll |\Phi|$ and $c_T^2 = 1 \pm 10^{-15}$ (multi–messenger bound), implemented as a hard prior on curvature–sector couplings (cf. Vol. I, Ch. 3).
- **(L3) Background bounds:** $|w_0 + 1| \lesssim 0.05$, $|w_a| \lesssim 0.3$, with the derived $H(z)$ matching SNe Ia+BAO+CC compilations at $\lesssim 1\%$ accuracy.
- **(L4) Growth and clustering:** $\mu(a,k) = 1 + \mathcal{O}(\delta)$ and $\Sigma(a,k) = 1 + \mathcal{O}(\delta)$ on linear/quasi–linear scales; predicted γ within current $f\sigma_8$ uncertainties.
- **(L5) Constant $G_{\rm eff}$:** $|\dot{G}/G| \lesssim 10^{-12}\,{\rm yr}^{-1}$, restricting $\dot{M}_{\rm eff}$.
- **(L6) Smooth hand–off:** The late–time solution is a continuous continuation of the reheated post–inflationary branch (Chap. 6) without triggering new instabilities in (ρ, χ^μ) sectors.

Summary. Ultra–long η relaxation provides a controlled, calculable alternative to Λ: it yields $w_\eta \simeq -1 + \delta(t)$ with small, correlated drifts in $H(z)$, a predictable offset in the growth index γ, and ISW/lensing ratios close to GR yet measurably distinct from a rigid constant. All effects are governed by the same spectral parameters $(V_\eta, \kappa_\eta, \Gamma_\eta; M_{\rm eff})$ that appear in the inflationary and reheating dynamics, enabling end–to–end constraints in Sec. 9.3.

9.3 Joint Fits and Cross–correlations

Objective. To confront the unified dark–sector hypothesis (Sec. 9.1–9.2) with data in a *single*, end–to–end analysis that exploits both one–point (background) and two–point (clustering/lensing) observables *and* their cross–correlations, while enforcing the consistency gates of the framework. The joint pipeline maps spectral parameters

$$\Theta_{\rm spec} \;=\; \big\{V_\eta,\ \kappa_\eta,\ \Gamma_\eta,\ M_{\rm eff}(\rho);\ \ell,\ \sigma/M,\ f_{\rm def}(z),\ \Pi_{\rm core}\big\}$$

(and any minimal nuisance set) to an *effective* late–time description $\{H(z),\, w(a),\, \mu(a,k),\, \Sigma(a,k),\, P_{\rm m}(k,z),\, c_T(a)\}$, from which all data vectors are forward–modeled.

9.3.1 Global data vector and forward models

We assemble a stacked data vector

$$\mathbf{d} \;=\; \Big[\underbrace{\mu_{\rm SN}(z_i)}_{\text{SNe Ia}},\ \underbrace{D_H(z_j), D_M(z_j)}_{\text{BAO}},\ \underbrace{H(z_k)}_{\text{CC}},\ \underbrace{f\sigma_8(z_\alpha)}_{\text{RSD}},\ \underbrace{C_\ell^{\gamma\gamma},\, C_\ell^{gg},\, C_\ell^{g\gamma}}_{\text{WL/galaxy}},\ \underbrace{C_\ell^{Tg}}_{\text{ISW–LSS}},\ \underbrace{C_\ell^{\phi\phi}}_{\text{CMB lens}},\ \underbrace{d_L^{\rm GW}(z_s)}_{\text{Sirens}},\ \underbrace{c_T}_{\text{GW speed}}\Big]^T, \tag{9.28}$$

with the following forward models (all under the effective metric $g_{\rm eff}$):

$$\mu_{\rm SN}(z) = 5\log_{10}\big[D_L^{\rm EM}(z)/10\,{\rm pc}\big], \qquad D_L^{\rm EM}(z) = (1+z)\int_0^z \frac{dz'}{H(z')}, \tag{9.29}$$

$$\{D_H, D_M\}_{\rm BAO} : D_H = c/H(z), \quad D_M = (1+z)D_A(z), \quad D_A(z) = \frac{1}{1+z}\int_0^z \frac{dz'}{H(z')}, \tag{9.30}$$

$$f\sigma_8(z) : \text{solve } D'' + \left(\frac{3}{a} + \frac{H'}{H}\right) D' - \frac{3}{2}\frac{\Omega_m(a)}{a^2}\mu(a, k\simeq k_{\rm RSD})D = 0, \tag{9.31}$$

$$C_\ell^{XY} : C_\ell^{XY} = \int_0^\infty \frac{dz}{H(z)}\, W_X(z)\, W_Y(z)\, \Sigma^p(a,k)\, \mu^q(a,k)\, P_{\rm m}\left(k = \frac{\ell+1/2}{\chi(z)}, z\right), \tag{9.32}$$

$$C_\ell^{Tg} :\ \text{ISW kernel } W_T \propto d(\Phi+\Psi)/dz, \quad W_g = b_g(z)\, n_g(z), \tag{9.33}$$

$$d_L^{\rm GW}(z) :\ d_L^{\rm GW}(z) = d_L^{\rm EM}(z)\, \exp\left[\frac{1}{2}\int_0^z \frac{\alpha_M(z')}{1+z'}\, dz'\right], \quad \alpha_M \equiv \frac{d\ln M_{\rm eff}^2}{d\ln a}, \tag{9.34}$$

$$c_T :\ c_T^2 = 1 + \delta c_T(\Theta_{\rm spec}), \qquad |\delta c_T| \le 10^{-15} \text{ (multi–messenger)}. \tag{9.35}$$

Here $X, Y \in \{\gamma, g, \phi\}$ denote shear, galaxy, and CMB–lensing fields; (p, q) are integers selecting the appropriate modified–gravity response (e.g. $p = 1, q = 0$ for lensing, $p = 0, q = 1$ for RSD). The matter power spectrum $P_{\rm m}(k, z)$ is obtained by evolving the primordial spectrum (Chap. 4) through the transfer function modified by μ, Σ and by the solitonic component:

$$P_{\rm m}(k,z) = \big[(1-f_{\rm sol})\,\delta_{\rm cdmb} + f_{\rm sol}\,\delta_{\rm sol}\big]^2 \;\Rightarrow\; P_{\rm m}(k,z) = T_{\rm eff}^2(k,z;\Theta_{\rm spec})\, P_{\rm prim}(k), \tag{9.36}$$

where $\delta_{\rm sol}$ encodes the linear response of the soliton population (bias $b_{\rm sol}(k, z)$, suppression scale $k_{\rm core} \sim \pi/\ell$) and $f_{\rm sol}(z)$ is the soliton mass fraction consistent with Sec. 5.3 bounds.

9.3.2 Statistical model, priors, and consistency gates

Assuming a near–Gaussian sampling distribution for the stacked vector,

$$-2\ln\mathcal{L}(\mathbf{d}\,|\,\Theta) \;=\; \big[\mathbf{d} - \mathbf{t}(\Theta)\big]^T \mathbf{C}^{-1}\big[\mathbf{d} - \mathbf{t}(\Theta)\big] + \ln\det(2\pi\mathbf{C}), \qquad \Theta \equiv \Theta_{\rm spec} \cup \Theta_{\rm nuis}, \tag{9.37}$$

with $\mathbf{t}(\Theta)$ the theory vector from Eqs. (9.29)–(9.36). The covariance $\mathbf{C}$ includes auto– and cross–blocks, e.g. $\mathrm{Cov}(C_\ell^{\gamma\gamma}, C_\ell^{g\gamma})$, $\mathrm{Cov}(C_\ell^{\gamma\gamma}, C_\ell^{\phi\phi})$, and $\mathrm{Cov}(C_\ell^{Tg}, C_\ell^{gg})$; where unavailable, we propagate survey–provided estimates or simulate them in a lognormal approximation.

Parameter hierarchy and priors.

$$\begin{aligned}\Theta_{\rm spec}:\ & \underbrace{(V_\eta,\kappa_\eta,\Gamma_\eta)}_{\text{late }\eta\text{ relaxation}},\ \underbrace{M_{\rm eff,0},\ \alpha_M(a)}_{\text{Planck–mass drift}},\ \underbrace{(\ell,\ \sigma/M,\ f_{\rm sol}(z),\ \Pi_{\rm core})}_{\text{soliton sector}},\\ \Theta_{\rm nuis}:\ & \{b_g(z),\ A_{\rm IA},\ \Delta z\text{ calibs},\ \Sigma_8\text{ prior},\ \dots\}.\end{aligned} \tag{9.38}$$

We impose *hard* consistency gates as priors: (i) $|\delta c_T| \le 10^{-15}$, (ii) $|\dot{G}/G| \le 10^{-12}\,{\rm yr}^{-1} \Rightarrow$ bound on α_M, (iii) stability: $0 < \mu, \Sigma \le 1+\epsilon$, $c_s^2 \in (0,1]$, (iv) astrophysical: σ/M and ℓ within Sec. 5.3 windows.

Posterior and sampling. $\mathcal{P}(\Theta|\mathbf{d}) \propto \mathcal{L}(\mathbf{d}|\Theta)\,\Pi(\Theta)$, sampled via nested sampling/tempered MCMC; for forecasting, the Fisher matrix

$$F_{ij} = \frac{\partial \mathbf{t}^T}{\partial \Theta_i}\,\mathbf{C}^{-1}\,\frac{\partial \mathbf{t}}{\partial \Theta_j}\bigg|_{\Theta=\Theta_\star} \tag{9.39}$$

gives marginalized errors and parameter correlations.

9.3.3 Key cross–correlations and discriminants

1. **ISW–LSS:** C_ℓ^{Tg} depends on the time derivative of $\Phi+\Psi$, thus on $w(a)$ and $\Sigma(a,k)$. The model predicts an *enhancement* $\Delta C_\ell^{Tg}/C_\ell^{Tg} \sim \mathcal{O}(\delta)$ for $\delta \equiv 1+w_0 > 0$, with scale–dependence tied to the soliton suppression scale $k_{\rm core}$ via $P_{\rm m}$.
2. **Shear–lensing–galaxy triangle:** The triplet $\{C_\ell^{\gamma\gamma}, C_\ell^{\phi\phi}, C_\ell^{g\gamma}\}$ tests the internal consistency of $\Sigma(a,k)$ and $P_{\rm m}$. Our gates require $\Sigma = 1+\mathcal{O}(\delta)$, implying that any observed deviation must correlate with $w(a)$ and *not* demand a large gravitational slip.
3. **RSD–lensing synergy:** Joint ($f\sigma_8$, $C_\ell^{\gamma\gamma}$) isolates μ vs. Σ; in our framework both are ≈ 1, so residual tensions should align with the predicted growth–index offset $\gamma - 0.55$ in Eq. (9.27).
4. **Standard sirens:** ($d_L^{\rm GW}$, $d_L^{\rm EM}$) ratio constrains $\alpha_M(a)$ by Eq. (9.34) *independently* of H_0. With c_T fixed to unity, sirens become a sharp probe of $M_{\rm eff}$ drift, directly cross–checking ISW and growth inferences.

9.3.4 Emulation and model reduction

To make the end–to–end fit tractable:

- Build a Gaussian–process emulator for $\{\mu,\Sigma\}(a,k)$ and $T_{\rm eff}(k,z)$ on a training grid in $\Theta_{\rm spec}$ obtained from the linear–response soliton solver and late–time η evolution.
- Use a two–tier compression: (i) map $\Theta_{\rm spec} \to \{w_0, w_a, \alpha_M^0, k_{\rm core}, f_{\rm sol}^0, \sigma/M\}$; (ii) feed these to a modified Boltzmann/LSS code to compute $\mathbf{t}(\Theta)$.
- Validate with end–to–end forward simulations at a sparse set of *anchor* points to control emulator bias.

9.3.5 Diagnostics, tensions, and stress–tests

We will report:

1. **Joint posterior corners** for $(w_0, w_a, \gamma, k_{\rm core}, f_{\rm sol}, \sigma/M, \alpha_M^0)$, with projected constraints and degeneracy rotations.
2. **Null tests** demanded by the gates: (i) $\Sigma - \mu \simeq 0$, (ii) $c_T - 1 \simeq 0$, (iii) $\alpha_M \simeq 0$; any significant deviation must be *coherently* reflected in ISW and siren sectors.
3. **Internal consistency** of LSS triangle (shear–lensing–galaxy) and ISW amplitude vs. background drift $\delta(a)$.
4. **Stress–tests** varying the small–scale modeling of solitons (changing $b_{\rm sol}$ or the core profile) to check the robustness of large–scale inferences.

Summary. Eq. (9.37) with the forward models (9.29)–(9.35) defines a practical, falsifiable route from spectral parameters to observations. The *same* $(V_\eta, \kappa_\eta, \Gamma_\eta; M_{\rm eff})$ that governed inflation and reheating are now tested at late times through $(w_0, w_a, \gamma, \alpha_M)$, while the soliton sector leaves controlled imprints via $(k_{\rm core}, f_{\rm sol}, \sigma/M)$. Cross–correlations (ISW–LSS, lensing triangles, sirens) supply the critical levers to distinguish this unified scenario from ΛCDM and generic dark–energy parametrizations.

Chapter 10

Inference Pipelines: CMB, LSS, Lensing, and GW

10.1 Datasets, Summary Statistics, and Forecasts

Scope. We assemble an end–to–end, survey–agnostic blueprint for inference on the spectral–cosmology parameters by combining Cosmic Microwave Background (CMB) temperature/polarization and lensing, Large–Scale Structure (LSS) clustering and redshift–space distortions (RSD), cosmic shear (weak lensing), and gravitational–wave (GW) probes (standard sirens, propagation constraints, and stochastic backgrounds). This section defines the *data vectors*, *summary statistics*, *covariance models*, and *forecast methodology* used in the joint pipeline. Mapping from spectral couplings to observables is developed in Sec. 16.2; priors and sampling strategies in Sec. 16.3.

10.1.1 Global data vector and survey placeholders

We collect measurements into a stacked vector

$$\mathbf{d} = \Big[\underbrace{\hat{C}_\ell^{TT},\, \hat{C}_\ell^{TE},\, \hat{C}_\ell^{EE},\, \hat{C}_\ell^{BB}}_{\text{CMB T/E/B}},\ \underbrace{\hat{C}_L^{\phi\phi}}_{\text{CMB lensing}},\ \underbrace{\hat{P}_\ell^{gg}(k),\ \hat{P}_\ell^{\mathrm{RSD}}(k)}_{\text{galaxy clustering \& RSD}},\ \underbrace{\hat{C}_\ell^{\gamma\gamma,\,ij}}_{\text{cosmic shear tomographic}},\ \underbrace{\hat{C}_\ell^{g\gamma,\,ij},\ \hat{C}_\ell^{\phi g},\ \hat{C}_\ell^{\phi\gamma}}_{\text{crosses}},\ \underbrace{\hat{\mu}_{\mathrm{SN}}(z),\ \hat{D}_{\mathrm{B}}}_{\text{background}} \tag{10.1}$$

Survey placeholders include ground/space CMB (Planck/ACT/SPT/Simons), spectroscopic LSS (DESI/Euclid), imaging lensing (DES/LSST/Euclid), pulsar timing arrays (PTA), and ground/space GW (LIGO/Virgo/KAGRA/ET/CE/LISA). Exact bandpowers, masks, and noise levels enter through experiment–specific windows; our formalism remains general.

10.1.2 CMB temperature/polarization and lensing

Pseudo-C_ℓ estimators and noise. For masked skies with fraction f_{sky}, we model the pseudo–spectra and their Gaussian covariance as

$$\langle \hat{C}_\ell^{XY} \rangle \approx \sum_{\ell'} M_{\ell\ell'}^{XY} C_{\ell'}^{XY}, \qquad \mathrm{Cov}(\hat{C}_\ell^{XY}, \hat{C}_{\ell'}^{WZ}) \approx \frac{\delta_{\ell\ell'}}{(2\ell+1) f_{\mathrm{sky}}} \Big[(C_\ell^{XW} + N_\ell^{XW})(C_\ell^{YZ} + N_\ell^{YZ}) + (X \leftrightarrow Y) \Big], \tag{10.2}$$

with $X, Y, W, Z \in \{T, E, B\}$, mode–coupling matrix $M_{\ell\ell'}$ from the mask, and instrument noise $N_\ell^{TT} = \Delta_T^2\, e^{\ell(\ell+1)\theta_{\mathrm{FWHM}}^2/8\ln 2}$, $N_\ell^{EE} = N_\ell^{BB} = 2\,\Delta_P^2\, e^{\ell(\ell+1)\theta_{\mathrm{FWHM}}^2/8\ln 2}$. CMB lensing bandpowers $\hat{C}_L^{\phi\phi}$ have reconstruction noise $N_L^{\phi\phi}$ from quadratic/iterative estimators; cross–covariances with T/E are included when forecasting.

Theory. Given the primordial spectra (Chap. 4), the lensed theory spectra follow from line–of–sight integrations on the effective metric g_{eff}, incorporating small deviations ($c_T \simeq 1 + \delta c_T$) and mild statistical anisotropy (when allowed by gates). Foregrounds and calibration nuisance parameters enter multiplicatively/additively with priors.

10.1.3 Galaxy clustering and RSD

3D power and multipoles. In redshift space,

$$P_s(k,\mu,z) = \Big[b_1(z) + f(z)\mu^2\Big]^2 P_{\rm m}(k,z)\,\mathcal{D}_{\rm FoG}(k,\mu,\sigma_v) \;+\; P_{\rm SN}, \tag{10.3}$$

with $\mu \equiv \hat{\mathbf{k}}\cdot\hat{\mathbf{n}}$, linear growth rate $f = d\ln D/d\ln a$, FoG damping $\mathcal{D}_{\rm FoG}$ (e.g. Gaussian with velocity dispersion σ_v), and shot noise $P_{\rm SN} = \bar{n}^{-1}$. Multipoles are

$$P_\ell(k) = \frac{2\ell+1}{2}\int_{-1}^{1} d\mu\, P_s(k,\mu)\,\mathcal{L}_\ell(\mu), \qquad \ell = 0,2,4. \tag{10.4}$$

The matter spectrum $P_{\rm m}$ carries the spectral–cosmology imprints via $\mu(a,k), \Sigma(a,k)$ (Sec. 16.2) and the soliton sector transfer $T_{\rm eff}(k,z)$.

Covariance. For a survey volume V and bin width Δk,

$$\mathrm{Cov}\big(P_\ell(k), P_{\ell'}(k')\big) \approx \frac{2(2\pi)^3}{V\,4\pi k^2\Delta k}\,\delta_{kk'}\,\mathcal{I}_{\ell\ell'}(k), \tag{10.5}$$

where $\mathcal{I}_{\ell\ell'}$ are angular integrals over μ including $(P_s + P_{\rm SN})^2$. Super–sample variance and window coupling can be added via response functions.

10.1.4 Cosmic shear tomography and cross–correlations

Shear spectra. For source bins i, j,

$$C_\ell^{\gamma\gamma,ij} = \int_0^{z_{\rm max}} \frac{dz}{H(z)}\,\frac{W_i(z)\,W_j(z)}{\chi^2(z)}\,\Sigma^2(a,k)\,P_{\rm m}\!\left(k = \frac{\ell+1/2}{\chi(z)}, z\right) \;+\; \delta_{ij}\frac{\sigma_\epsilon^2}{\bar{n}_i}, \tag{10.6}$$

with lensing kernels W_i, intrinsic shape noise σ_ϵ, and $\Sigma(a,k)$ the light–deflection response. Intrinsic alignments (IA) add templates $C_\ell^{\rm IA}$ with amplitude $A_{\rm IA}$ and redshift slope.

Crosses with galaxies and CMB lensing. Galaxy–shear and lensing–shear spectra read

$$C_\ell^{g\gamma,ij} = \int \frac{dz}{H}\,\frac{W_g(z)\,W_j(z)}{\chi^2}\,\Sigma\,\mu\,P_{\rm m}, \qquad C_\ell^{\phi\gamma,i} = \int \frac{dz}{H}\,\frac{W_\phi(z)\,W_i(z)}{\chi^2}\,\Sigma\,P_{\rm m}, \tag{10.7}$$

with galaxy kernel $W_g = b_g\,n_g$ and lensing potential kernel W_ϕ. These crosses are powerful discriminants of μ vs. Σ.

10.1.5 Background probes: SNe Ia and BAO

SNe Ia. Distance moduli obey

$$\mu_{\rm SN}(z) = 5\log_{10}\big[D_L^{\rm EM}(z)/10\,{\rm pc}\big], \qquad D_L^{\rm EM}(z) = (1+z)\int_0^z \frac{dz'}{H(z')}, \tag{10.8}$$

with nuisance calibration parameters (M_B, α, β) marginalised.

BAO. We fit $D_H = c/H(z)$, $D_M = (1+z)D_A(z)$ and/or the volume distance D_V, with a covariance including reconstruction systematics and redshift errors.

10.1.6 Gravitational–wave probes

Standard sirens. Luminosity distances from compact–binary mergers provide

$$d_L^{\rm GW}(z) = d_L^{\rm EM}(z)\,\exp\!\left[\frac{1}{2}\int_0^z \frac{\alpha_M(z')}{1+z'}\,dz'\right], \qquad \alpha_M \equiv \frac{d\ln M_{\rm eff}^2}{d\ln a}, \tag{10.9}$$

with per–event variance $\sigma_{d_L}^2 = \sigma_{\rm inst}^2 + \sigma_{\rm lens}^2(z) + \sigma_{\rm pv}^2(z)$. For catalogues $\{z_s^i, d_{L,i}^{\rm GW}\}$, the likelihood is Gaussian in $\ln d_L$ including selection functions.

Propagation speed and friction. Multi–messenger timing constrains $|c_T - 1|$, entering CMB/GW kernels and enforcing $|\delta c_T| \ll 1$. The amplitude damping probes $\alpha_M(a)$ independently of H_0.

PTA/Stochastic backgrounds. For energy density spectrum $\Omega_{\rm GW}(f)$,

$$\Omega_{\rm GW}(f) = \frac{2\pi^2}{3H_0^2} f^3 S_h(f), \qquad \mathrm{SNR}^2 = 2T \sum_{a<b} \int df \, \frac{\Gamma_{ab}^2(f)\, S_h^2(f)}{P_a(f) P_b(f)}, \tag{10.10}$$

with S_h the strain PSD, Γ_{ab} the Hellings–Downs overlap, P_a the noise PSD, and T the timespan. Forecasts marginalise timing model parameters and clock/ephemeris systematics.

10.1.7 Joint likelihood and covariance structure

Gaussian core. We adopt a block–Gaussian likelihood for the stacked vector:

$$-2\ln \mathcal{L}(\mathbf{d}\,|\,\Theta) = \big[\mathbf{d} - \mathbf{t}(\Theta)\big]^T \mathbf{C}^{-1} \big[\mathbf{d} - \mathbf{t}(\Theta)\big] + \ln\det(2\pi\mathbf{C}), \tag{10.11}$$

where $\mathbf{t}(\Theta)$ is the theory vector computed from Eqs. (10.2)–(10.10) and $\mathbf{C}$ includes auto– and cross–covariances (e.g. $\mathrm{Cov}(C_\ell^{\gamma\gamma}, C_\ell^{\phi\phi})$, $\mathrm{Cov}(C_\ell^{Tg}, C_\ell^{gg})$). Non–Gaussian corrections (trispectra, super–sample variance) are added via semi–analytic templates or simulation–calibrated response.

Nuisance parameters. We include: calibration and beam parameters (CMB), foreground amplitudes/spectral indices, galaxy bias and stochasticity, FoG σ_v, IA amplitude and slope, photo–z shifts Δz_i, multiplicative shear bias m_i, siren selection and inclination priors, PTA clock/ephemeris terms. Priors reflect survey systematics budgets.

10.1.8 Forecast methodology

Fisher forecasts. Around a fiducial $\Theta_\star$,

$$F_{ij} = \frac{\partial \mathbf{t}^T}{\partial \Theta_i} \mathbf{C}^{-1} \frac{\partial \mathbf{t}}{\partial \Theta_j}\bigg|_{\Theta_\star} + \frac{1}{2}\,\mathrm{Tr}\left[\mathbf{C}^{-1} \frac{\partial \mathbf{C}}{\partial \Theta_i} \mathbf{C}^{-1} \frac{\partial \mathbf{C}}{\partial \Theta_j}\right], \qquad \sigma(\Theta_i) \geq \sqrt{(F^{-1})_{ii}}. \tag{10.12}$$

Derivatives are computed by stable two–sided differences or automatic differentiation of the Boltzmann/LSS kernels. We propagate survey masks via $M_{\ell\ell'}$ and k–space windows.

Compression and emulation. For high–dimensional spectra we apply MOPED/KL compression: $\mathbf{y} = \mathbf{W}^T\mathbf{d}$ with $\mathbf{W} \propto \mathbf{C}^{-1}\partial\mathbf{t}/\partial\Theta$. Where theory evaluations are costly (e.g. soliton transfer $T_{\rm eff}$), we deploy Gaussian–process emulators trained on a Latin–hypercube in parameter space; emulator uncertainty is injected into $\mathbf{C}$.

Model selection. Bayesian evidence $\mathcal{Z} = \int d\Theta\, \mathcal{L}\Pi$ (nested sampling) compares the unified spectral model against ΛCDM or wCDM, reporting $\Delta \ln \mathcal{Z}$ and Occam penalties.

10.1.9 Systematics model and validation

We include foreground residual templates (CMB), scale–cuts for nonlinearity and baryons, photometric–z and multiplicative shear priors (lensing), fiber collisions and completeness (clustering), selection and weak–lensing magnification (sirens), and clock/ephemeris marginalisation (PTA). Validation proceeds via:

1. End–to–end mock catalogues that preserve cross–covariances across probes.
2. Recovery tests of injected Θ, including stress–tests under mis–specified nuisance.
3. Consistency *nulls*: internal CMB splits, shear B–modes, lensing–ISW sign, RSD–lensing growth consistency.

Summary. Eqs. (10.1)–(10.12) define a modular, cross–correlated inference scaffold capable of testing the spectral–cosmology framework against present and upcoming data. Crucially, GW standard sirens and CMB/LSS lensing triangles provide orthogonal levers on $\alpha_M(a)$, $\mu(a,k)$, $\Sigma(a,k)$, while PTA bounds inform early–time spectral dynamics through $\Omega_{\rm GW}(f)$. This integrated pipeline is the quantitative backbone for the commitments articulated in Volume II.

10.2 Parameter Map from Spectral Couplings to Observables

Objective. We construct an explicit, differentiable map

$$\Theta_{\rm spec} \longrightarrow \Big\{\mathcal{P}_\zeta(k,\theta),\ \mathcal{P}_T(k)\Big\} \longrightarrow \Big\{\mu(a,k),\ \Sigma(a,k),\ T_{\rm eff}(k,z)\Big\} \longrightarrow \mathbf{t}(\Theta)\,,$$

where $\Theta_{\rm spec}$ are spectral (micro)–parameters, $\{\mathcal{P}_\zeta, \mathcal{P}_T\}$ the primordial spectra (Chap. 4), $\{\mu, \Sigma\}$ the linear response functions for clustering and lensing, $T_{\rm eff}$ the transfer from the dark–soliton sector, and $\mathbf{t}(\Theta)$ the joint data vector in Eq. (10.1).

10.2.1 Parameter vector and staging of the map

We organise the parameter space into *nodes* of the computational graph:

$$\Theta_{\rm spec} \equiv \underbrace{\{V_0,\ \eta_\star,\ \kappa_{\eta,0},\ \Gamma_{\eta,0}\}}_{\eta\text{-wave inflation}} \cup \underbrace{\{M_{\rm eff,0},\ \alpha_M(a),\ \delta c_T(a)\}}_{\text{curvature/propagation}} \cup \underbrace{\{A_\chi,\ \hat{\chi}^{(0)}\}}_{\text{stat. anisotropy seed}} \cup \underbrace{\{\theta_{\rm sol}\}}_{\text{dark soliton sector}}$$
$$\cup \underbrace{\{\Omega_b h^2,\ \Omega_c h^2,\ \tau,\ T_{\rm CMB}, \dots\}}_{\text{baseline background}}. \tag{10.13}$$

The soliton block $\theta_{\rm sol}$ controls the small–scale transfer (core scale ℓ, effective sound speed $c_{\rm eff}$, self–interaction σ/m, etc.). Time dependences $\alpha_M(a) \equiv d\ln M_{\rm eff}^2/d\ln a$ and $\delta c_T(a) \equiv c_T^2(a) - 1$ are represented by low–order splines or principal–component amplitudes.

10.2.2 Primordial node: $\mathcal{P}_\zeta$, $\mathcal{P}_T$, n_s, r, g_*, $f_{\rm NL}$

Given the slow parameters (Sec. **??**, **??**),

$$\mathcal{P}_\zeta(k,\theta) = \frac{H^2}{8\pi^2\, M_{\rm eff}^2\, \epsilon_\eta\, c_s(\theta)}\bigg|_{k=aH/c_s}, \quad n_s - 1 \simeq -2\epsilon_\eta - \eta_\eta - s_c, \quad r \simeq 16\,\epsilon_\eta \frac{c_s}{c_T}\,. \tag{10.14}$$

Statistical anisotropy is parameterised as

$$\mathcal{P}_\zeta(k,\theta) = \mathcal{P}_0(k)\left[1 + g_*(k)\,(\hat{\mathbf{k}}\cdot\hat{\chi}^{(0)})^2 + \dots\right], \qquad g_*(k) \;\approx\; \mathcal{C}_\chi\, \frac{c_\parallel - c_\perp}{c_s}\bigg|_k, \tag{10.15}$$

where $\mathcal{C}_\chi$ is a calculable geometric factor from the director–ζ mixing (Sec. 4.3). The leading non–Gaussianity scales as

$$f_{\rm NL}^{\rm eff}(k_i,\theta_i) \;\approx\; \underbrace{\frac{1-c_s^2}{c_s^2}}_{\text{sound–speed}} + \underbrace{\kappa_{\rm mix}\,\Xi_{\chi\zeta}(k_i,\theta_i)}_{\text{anisotropic mixing}} + \mathcal{O}(\text{higher deriv.}), \tag{10.16}$$

with $\kappa_{\rm mix}$ a function of $\{A_\chi, \eta_\star, \kappa_{\eta,0}\}$ and $\Xi_{\chi\zeta}$ a known shape form factor.

Tensor power and consistency reads

$$\mathcal{P}_T(k) = \frac{2H^2}{\pi^2 M_{\rm eff}^2 c_T}\bigg|_{k=aH/c_T}, \qquad r = \frac{\mathcal{P}_T}{\mathcal{P}_\zeta} \simeq 16\,\epsilon_\eta\,\frac{c_s}{c_T}\,. \tag{10.17}$$

10.2.3 Linear response node: $\mu(a,k)$ and $\Sigma(a,k)$

On sub–horizon scales, the modified Poisson and lensing relations are encoded by

$$-k^2\Psi = 4\pi G a^2\,\mu(a,k)\,\rho_m\Delta, \qquad -k^2(\Phi+\Psi) = 8\pi G a^2\,\Sigma(a,k)\,\rho_m\Delta, \tag{10.18}$$

with $\mu = \Sigma = 1$ in GR. In the spectral curvature sector we write to first nontrivial order

$$\mu(a,k) = 1 + \sum_{n=0}^{N} \beta_{\mu,n}(a)\,\frac{k^{2n}}{k_\star^{2n}}, \qquad \Sigma(a,k) = 1 + \sum_{n=0}^{N} \beta_{\Sigma,n}(a)\,\frac{k^{2n}}{k_\star^{2n}}, \tag{10.19}$$

where the coefficients $\beta_{\mu,\Sigma}$ are *linear functionals* of the spectral couplings $\mathcal{C}$ that enter the emergent–gravity Lagrangian (Vol. I, Ch. 3):

$$\beta_{\mu,n}(a) = \int da'\,\mathcal{K}_{\mu,n}(a,a')\,\mathcal{C}(a'), \qquad \beta_{\Sigma,n}(a) = \int da'\,\mathcal{K}_{\Sigma,n}(a,a')\,\mathcal{C}(a'). \tag{10.20}$$

In the quasi–static limit a convenient closed form is

$$\mu(a,k) = 1 + \frac{\alpha_B^2(a)}{c_s^2(a)}\,\frac{k^2}{k^2 + k_V^2(a)}, \qquad \Sigma(a,k) = \mu(a,k)\left[1 + \frac{\alpha_T(a)}{2}\right], \tag{10.21}$$

where α_B (braiding), $\alpha_T = \delta c_T$, c_s^2 are functions of $\Theta_{\rm spec}$ and k_V a transition scale; Eq. (10.21) is the default emulator target used in forecasts.

The linear growth obeys

$$D'' + \left[2 + \frac{H'}{H}\right] D' - \frac{3}{2}\,\Omega_m(a)\,\mu(a,k)\,D = 0, \qquad P_{\rm m}(k,z) = D^2(a)\,T_{\rm eff}^2(k,a)\,P_{\rm prim}(k), \tag{10.22}$$

with $P_{\rm prim}(k) \propto \mathcal{P}_\zeta(k)$ and $T_{\rm eff}$ defined next.

10.2.4 Dark–soliton node: transfer $T_{\rm eff}(k,a)$ and σ/m

Coherent ρ–structures (Chap. 5) imprint a small–scale transfer. A flexible, physics–motivated template is

$$T_{\rm eff}(k,a) = \left[1 + \left(\frac{k}{k_J(a)}\right)^{2n}\right]^{-m/2} \exp\left[-\left(\frac{k}{k_d(a)}\right)^{p}\right], \qquad k_J(a) = \frac{aH}{c_{\rm eff}(a)}\sqrt{\frac{3}{2}\,\Omega_m(a)\,\mu(a,k_J)}. \tag{10.23}$$

Here k_J plays the role of a Jeans–like scale set by the effective soliton sound speed $c_{\rm eff}$, and k_d accounts for diffusive damping during reheating/thermalisation; (n,m,p) are integers of order unity fixed by the sector's EFT. The *self–interaction* cross section per mass,

$$\frac{\sigma}{m} \simeq \frac{4\pi\,a_{\rm eff}^2}{m} \sim \frac{\lambda_{\rm spec}^2}{4\pi\,m_{\rm sol}^3}, \tag{10.24}$$

maps the microscopic spectral coupling $\lambda_{\rm spec}$ and soliton mass scale $m_{\rm sol}$ to astrophysical constraints $(\sigma/m)\big|_v$ (Sec. 5.3); priors enforce σ/m within cluster/dwarf bounds.

10.2.5 Propagation node: $c_T(a)$, $\alpha_M(a)$ and GW distances

The GW speed and friction are given by

$$c_T^2(a) = 1 + \delta c_T(a), \qquad \alpha_M(a) = \frac{d\ln M_{\rm eff}^2}{d\ln a}, \tag{10.25}$$

feeding into the standard–siren luminosity distance via Eq. (10.9) and into the tensor spectrum through Eq. (10.17). Multi–messenger priors strongly constrain $\delta c_T(a)$ at $z \lesssim \mathcal{O}(0.1-1)$.

10.2.6 From nodes to spectra and correlations

CMB spectra follow from line–of–sight integrals with modified sources:

$$C_\ell^{XY} = \int d\ln k\; \mathcal{P}_\zeta(k)\, \Delta_\ell^X(k)\, \Delta_\ell^Y(k) \;+\; \int d\ln k\; \mathcal{P}_T(k)\, \Delta_{\ell,T}^X(k)\, \Delta_{\ell,T}^Y(k), \tag{10.26}$$

$$\Delta_\ell^X(k) = \int_0^{\eta_0} d\eta\; S_X(k,\eta;\mu,\Sigma,\alpha_M)\, j_\ell[k(\eta_0-\eta)]\,, \tag{10.27}$$

where the source functions S_X carry the $\{\mu,\Sigma\}$ dependence (ISW, lensing, Doppler) and α_M via background/propagation. LSS, lensing, and cross spectra are given by Eqs. (10.3)–(10.7) with $P_{\rm m}$ from Eq. (10.22).

10.2.7 Differentiation rules for forecasts and samplers

For Fisher derivatives (Eq. 10.12) and gradient–based samplers we use the chain rule through the nodes:

$$\frac{\partial \mathbf{t}}{\partial \Theta_i} = \sum_{\pi\in\{\mathcal{P}_\zeta,\mathcal{P}_T,\mu,\Sigma,T_{\rm eff},H,\ldots\}} \frac{\partial \mathbf{t}}{\partial \pi}\cdot\frac{\partial \pi}{\partial \Theta_i}\,. \tag{10.28}$$

Representative closed–form pieces include

$$\frac{\partial \ln \mathcal{P}_\zeta}{\partial \ln \epsilon_\eta} = -1, \quad \frac{\partial \ln \mathcal{P}_\zeta}{\partial \ln M_{\rm eff}^2} = -1, \quad \frac{\partial \ln \mathcal{P}_\zeta}{\partial \ln c_s} = -1, \tag{10.29}$$

$$\frac{\partial \ln P_{\rm m}}{\partial \mu(a,k)} = \mathcal{G}(a,k) \quad \text{with} \quad \mathcal{G}(a,k) \equiv \frac{\partial \ln D^2}{\partial \mu}\,, \qquad \frac{\partial \ln P_{\rm m}}{\partial \ln T_{\rm eff}} = 2, \tag{10.30}$$

$$\frac{\partial \ln d_L^{\rm GW}}{\partial \alpha_M(a)} = \frac{1}{2}\,\frac{\Theta(z-a)}{1+z}\,, \qquad \frac{\partial \ln \mathcal{P}_T}{\partial \ln c_T} = -1. \tag{10.31}$$

Anisotropy derivatives use $\partial\mathcal{P}_\zeta/\partial g_* = \mathcal{P}_0(k)\,(\hat{\mathbf{k}}\cdot\hat{\chi}^{(0)})^2$ and Fisher–average over angles with the survey window.

10.2.8 Consistency–gate priors (hard/soft)

The mapping is constrained by the gates (Vol. I; Chap. 4):

$$\mathsf{K} \succ 0 \quad \text{(no ghosts)}, \qquad 0 < c_s^2(\theta), c_T^2 \le 1 \quad \text{(causality)}, \qquad \sigma_{\rm diss} \ge 0 \quad \text{(arrow)}, \qquad |g_*| \ll 1,\ |\,\delta c_T\,| \ll 1 \text{ at late times.} \tag{10.32}$$

We implement *hard* priors for positivity/causality and *soft* Gaussian priors for late–time δc_T, g_*, and σ/m.

10.2.9 Identifiability and degeneracy directions

CMB $TT/TE/EE$ chiefly constrain $(n_s,\ \omega_b,\ \omega_c)$ and the combination $A_s e^{-2\tau}$; low–ℓ polarization breaks (A_s,τ). The pair $(r,\ c_T)$ is constrained by B–modes and sirens; $\alpha_M(a)$ is probed by siren distances and CMB lensing amplitude. LSS $P_\ell(k)$ and shear tomographic spectra constrain μ and Σ in complementary ways; their degeneracy with $T_{\rm eff}$ is broken by small–scale cuts, halo statistics, and astrophysical σ/m priors. The anisotropy amplitude g_* is limited by CMB large–scale isotropy and by LSS multipole–mixing tests.

Summary. Eqs. (10.14)–(10.31) provide a transparent ladder from spectral couplings to the multi–probe observables used in Sec. 16.1. The map is modular (node–wise), differentiable (forecast–ready), and guarded by the theory's consistency gates (10.32), enabling principled joint fits in Sec. 16.3.

10.3 Priors as Consistency Gates and Sampling Strategies

Philosophy. In the spectral framework, *consistency gates* derived from the theory—stability, causality, gauge/topological admissibility, dissipative arrow, and numerical reliability—play the role of *structural priors*. We use them to constrain the parameter space before confronting data, thereby avoiding posterior mass in unphysical regions and improving sampler efficiency.

10.3.1 Gate-informed priors: hard constraints and soft penalties

We partition priors into *hard* (support-restricting) and *soft* (penalizing) components. Let Θ denote the full parameter vector defined in Eq. (10.13).

Hard gate priors (support).

$$\begin{aligned}\Pi_{\rm hard}(\Theta) = \mathbf{1}[\ \mathsf{K}(\Theta) \succ 0\ ] \cdot \mathbf{1}\big[\ 0 < c_s^2(\theta,\Theta) \le 1\ \big] \cdot \mathbf{1}\big[\ 0 < c_T^2(a,\Theta) \le 1\ \big] \\ \cdot\ \mathbf{1}[\ \sigma_{\rm diss}(\Theta) \ge 0\ ] \cdot \mathbf{1}[\ \Theta \text{ satisfies gauge/topology charts (Vol. I)}\ ],\end{aligned} \tag{10.33}$$

where K is the time-kinetic block (no-ghost), c_s^2 and c_T^2 ensure causal cones inside $g_{\rm eff}$, and $\sigma_{\rm diss} \ge 0$ enforces the entropy/arrow gate. These indicators can be implemented either by rejection or by *reparametrization* to an unconstrained space, e.g.

$$c_T^2 = \sigma(\tilde{c}_T) \equiv \frac{1}{1+e^{-\tilde{c}_T}}, \qquad c_s^2 = \sigma(\tilde{c}_s), \qquad M_{\rm eff}^2 = e^{\tilde{M}},$$

with $\tilde{c}_T, \tilde{c}_s, \tilde{M} \in \mathbb{R}$.

Soft gate priors (penalties). For late-time GW bounds and statistical isotropy:

$$\begin{aligned}\ln \Pi_{\rm soft}(\Theta) = &-\frac{1}{2}\sum_{z_i}\frac{\big[\delta c_T(z_i;\Theta)\big]^2}{\sigma_{c_T}^2(z_i)} - \frac{1}{2}\frac{g_*^2(\Theta)}{\sigma_{g_*}^2} - \frac{1}{2}\frac{\big[\alpha_M(z_{\rm low};\Theta)\big]^2}{\sigma_{\alpha_M}^2} \\ &- \frac{1}{2}\frac{\big[(\sigma/m)(v_\star;\Theta) - (\sigma/m)_\star\big]^2}{\Sigma_{\sigma/m}^2} - \frac{1}{2}\sum_n \frac{\beta_{\mu,n}^2}{\Sigma_{\beta_{\mu,n}}^2} - \frac{1}{2}\sum_n \frac{\beta_{\Sigma,n}^2}{\Sigma_{\beta_{\Sigma,n}}^2}.\end{aligned} \tag{10.34}$$

Here $\sigma_{c_T}(z)$ encodes multi–messenger limits, σ_{g_*} CMB isotropy bounds, σ_{α_M} siren/lensing calibration, $(\sigma/m)_\star$ astrophysical target with tolerance $\Sigma_{\sigma/m}$, and β-coefficients regularize excessive scale dependence in μ, Σ.

Full prior.

$$\Pi(\Theta) \ = \ \Pi_{\rm hard}(\Theta) \times \Pi_{\rm soft}(\Theta) \times \Pi_{\rm base}(\Theta), \tag{10.35}$$

where $\Pi_{\rm base}$ includes wide, weakly informative priors on standard background parameters, calibration factors, and nuisance fields.

10.3.2 Likelihood structure and emulator uncertainty

Given data vector $\mathbf{d}$ and theory vector $\mathbf{t}(\Theta)$ [Eq. (10.1)], we adopt a Gaussian likelihood with augmented covariance,

$$\mathcal{L}(\mathbf{d}|\Theta) \ \propto\ \exp\Big[-\tfrac{1}{2}\,\Delta^T\Big(C_{\rm stat} + C_{\rm sys} + C_{\rm emu}(\Theta)\Big)^{-1}\Delta\Big], \quad \Delta \equiv \mathbf{d} - \mathbf{t}(\Theta). \tag{10.36}$$

The emulator error covariance $C_{\rm emu}$ is propagated from the surrogate training residuals and *added* to statistical/systematic covariances, ensuring conservative posteriors. Cross–probe blocks are retained for CMB–lensing, lensing–clustering, and PTA–LSS correlations.

10.3.3 Sampling strategies: HMC/NUTS, nested sampling, and tempering

Gradient–based (HMC/NUTS). Using the differentiable map of Sec. 16.2 we sample the posterior $p(\Theta|\mathbf{d}) \propto \mathcal{L}(\mathbf{d}|\Theta)\,\Pi(\Theta)$ with Hamiltonian Monte Carlo (HMC) or the No–U–Turn Sampler (NUTS):

- *Reparametrization to unconstrained space* (log/scaled logits) for all hard–gate variables.
- *Dual averaging warm–up* to adapt step size; *dense mass–matrix* adaptation to decorrelate parameters.
- *Diagnostics:* $\widehat{R}<1.01$, effective sample size (ESS) per dimension, energy Bayesian fraction of missing information (E-BFMI), rank plots.

Nested sampling (evidence and multi–modality). For model selection (e.g. ΛCDM vs. spectral cosmology) and multi–modal posteriors we use dynamic nested sampling to obtain the evidence Z and robust tails. This is complemented by *parallel tempering* or *differential–evolution MCMC* when strong mode isolation is suspected.

Likelihood–free (simulation–based) inference. In non–Gaussian summary spaces (e.g. Minkowski functionals, peak counts), we adopt neural posterior estimation (normalizing flows) with simulation–based calibration. Summaries are compressed using score–compression where gradients are available; otherwise via information–maximizing neural compression.

10.3.4 Nuisance hierarchy and hierarchical priors

We include calibration parameters (absolute gain, beam, photometric redshifts), astrophysical foreground amplitudes, and shear/IA (intrinsic alignment) nuisances as a hierarchical block ϕ with priors informed by external calibration:

$$\Pi(\Theta, \phi) = \Pi(\Theta) \times \Pi(\phi|\text{calib}).$$

Marginalization is performed jointly in HMC/NUTS (if low–dimensional) or analytically when Gaussian. For shear/IA we employ TATT–style parameterizations with wide priors.

10.3.5 Reparameterizations and degeneracy control

To reduce curvature in the posterior:

- Use $(A_s e^{-2\tau},\ \tau)$ instead of (A_s, τ).
- Replace highly correlated (μ, Σ) spline nodes by principal components with orthonormal priors.
- Parameterize anisotropy by amplitude–direction $(g_*,\ \hat{\chi}^{(0)})$ with a von Mises–Fisher prior on $\hat{\chi}^{(0)}$.

10.3.6 Validation, coverage, and posterior–predictive checks

Mock–data validation.

1. Draw $\Theta_{\text{true}} \sim \Pi(\Theta)$, simulate $\mathbf{d}_{\text{mock}}$.
2. Fit $\mathbf{d}_{\text{mock}}$ to obtain $p(\Theta|\mathbf{d}_{\text{mock}})$.
3. Perform *simulation–based calibration*: rank histograms for each parameter should be uniform.

Posterior–predictive checks. Draw Θ from the posterior and compute replicated data $\mathbf{d}^{\text{rep}}$; test discrepancy statistics (e.g. χ^2, skewness of LSS statistics). Large systematic deviations indicate emulator mis–specification or missing systematics.

10.3.7 End-to-end algorithm (schematic)

1. **Gate compilation:** implement Π_{hard} via reparameterization; set Π_{soft} scales $(\sigma_{c_T}, \sigma_{g_*}, \Sigma_{\sigma/m})$.
2. **Emulators:** train node-wise surrogates for $\{\mathcal{P}_\zeta, \mathcal{P}_T, \mu, \Sigma, T_{\text{eff}}\}$; quantify C_{emu}.
3. **Likelihood wiring:** assemble multi–probe $\mathcal{L}$ with cross–covariances and nuisance hierarchy.
4. **Warm–up:** run NUTS with adaptation (reparametrized Θ); monitor $\widehat{R}$, ESS, E-BFMI.
5. **Production:** sample posterior; if multi–modal, switch to nested sampling/tempering.
6. **Validation:** SBC and posterior–predictive checks; iterate emulator retraining if needed.
7. **Delivery:** marginalized posteriors, evidences, and gate-satisfaction audit for posterior mass.

Summary. Consistency gates are elevated to *first-class priors* that sculpt the viable parameter region. Combined with differentiable mapping and robust samplers (NUTS/nested/SBI), this yields an inference pipeline that is both *physically faithful* and *computationally efficient*, ready for the joint analyses in Chap. 9 and for model comparison against ΛCDM.

Chapter 11

Numerical Protocols for Spectral Cosmology

11.1 Background and Perturbation Integrators

Scope. We present robust numerical schemes for (i) the *background* dynamics driven by the spectral–viscosity condensate $\bar{\eta}(t)$, the state–dependent Planck scale $M_{\rm eff}(\rho)$, and the coarse–grained fluids produced at reheating; and (ii) the *linear perturbations*—scalar curvature mode ζ (with anisotropic $c_s(\theta)$) and tensor mode h_{ij} (with $c_T \simeq 1 + \delta c_T$). The protocols are designed to (a) handle stiffness across slow–roll, exit, and reheating; (b) enforce the Friedmann constraint as an index–1 DAE; (c) preserve canonical normalizations (Wronskian constraints) for mode functions; and (d) furnish differentiable outputs and sensitivities for inference (Chap. 10).

11.1.1 Background system as an index–1 DAE

We work in the aether frame u^μ and adopt e–fold time $N \equiv \ln a$ as the independent variable. Let

$$\mathbf{y}(N) \equiv \left(\bar{\eta},\ Y_\eta \equiv \frac{d\bar{\eta}}{dN},\ H,\ \rho_r,\ \rho_m,\ \rho_{\rm curv},\ \dots\right)^T,$$

where ρ_r, ρ_m are radiation/matter energy densities (possibly zero before reheating), and $\rho_{\rm curv}$ is the coarse–grained curvature contribution. The governing equations are

$$\frac{d\bar{\eta}}{dN} = Y_\eta, \tag{11.1a}$$

$$\frac{dY_\eta}{dN} = -\left(3 + \frac{1}{H}\frac{d\ln\kappa_\eta}{dt} + \frac{\Gamma_\eta}{\kappa_\eta H}\right) Y_\eta - \frac{V_\eta'(\bar{\eta})}{\kappa_\eta H^2}, \tag{11.1b}$$

$$\frac{d\rho_r}{dN} = -4\rho_r + \frac{\Gamma_{\eta\to r}}{H}\,\frac{\kappa_\eta H^2 Y_\eta^2}{1}, \tag{11.1c}$$

$$\frac{d\rho_m}{dN} = -3\rho_m + \frac{\Gamma_{\eta\to m}}{H}\,\kappa_\eta H^2 Y_\eta^2, \tag{11.1d}$$

$$0 = \Phi_F(\mathbf{y}) \ \equiv\ 3M_{\rm eff}^2(\rho)\,H^2 - \rho_{\rm tot}(\mathbf{y}), \tag{11.1e}$$

with $\rho_{\rm tot} = \rho_\eta + \rho_r + \rho_m + \rho_{\rm curv} + \cdots$ and $\rho_\eta = \frac{1}{2}\kappa_\eta H^2 Y_\eta^2 + V_\eta(\bar{\eta})$, $p_\eta = \frac{1}{2}\kappa_\eta H^2 Y_\eta^2 - V_\eta(\bar{\eta})$. The last line is the Friedmann algebraic constraint, making (11.1) an *index–1* DAE. We solve it with **implicit** methods (Radau–IIA or variable–order BDF) using Newton iterations that *project* back to $\Phi_F = 0$ at each step (see below).

Projection to the constraint manifold. Given a trial step $\mathbf{y}^{\rm tr}$, we correct H via

$$H^{\rm new} \ \leftarrow\ \sqrt{\frac{\rho_{\rm tot}(\mathbf{y}^{\rm tr})}{3M_{\rm eff}^2(\rho^{\rm tr})}}, \qquad \epsilon_F \equiv \frac{|\Phi_F|}{3M_{\rm eff}^2 H^2} \ \overset{!}{<}\ \varepsilon_{\rm Ftol},$$

with $\varepsilon_{\rm Ftol} \sim 10^{-10} - 10^{-8}$. Optionally, perform a *Gauss–Newton* update on $(H, \bar{\eta})$ using $\partial\Phi_F/\partial(H, \bar{\eta})$ to accelerate convergence.

11.1.2 Stiffness detection and IMEX selection

We monitor stiffness through the spectral radius of the Jacobian $J = \partial\mathbf{f}/\partial\mathbf{y}$ (here $\mathbf{f}$ are the ODE parts). Define

$$\mathcal{S} \;\equiv\; \rho(J)\; h, \qquad h = \Delta N \text{ (current step)},$$

and switch schemes by thresholds: $\mathcal{S} < 1$: explicit ESDIRK–33 with embedded error; $1 \le \mathcal{S} \le 10$: diagonally–implicit (SDIRK–23); $\mathcal{S} > 10$: Radau–IIA–5 or BDF(2–5). Alternatively, use an **IMEX** split of $\mathbf{f} = \mathbf{f}_{\rm stiff} + \mathbf{f}_{\rm non}$, keeping Γ_η–damping and rapid $M_{\rm eff}$–drift in the implicit part.

Adaptive control and positivity. We adopt PI–controllers for the step size h:

$$h_{n+1} = h_n \left(\frac{\varepsilon_{\rm tol}}{{\rm err}_n}\right)^{\alpha} \left(\frac{{\rm err}_{n-1}}{\varepsilon_{\rm tol}}\right)^{\beta}, \quad (\alpha, \beta) = (0.7, 0.4),$$

with normed local error err (weighted RMS). Enforce $\rho_i \ge 0$ and $H > 0$ by exponential parameterizations, e.g. $\rho_i = e^{\tilde{\rho}_i}$, $H = e^{\tilde{H}}$, solved in the lifted variables.

11.1.3 Event handling: end of inflation and reheating milestones

We bracket and locate events by root finding (Hermite interpolation):

- *End of inflation:* $\epsilon_\eta(N_\star) \equiv -\dot{H}/H^2 = 1$. Reduce step, locate $N_\star$, then switch to a post–inflation integration profile (potentially different tolerances).
- *Reheating completion:* $f_r \equiv \rho_r/\rho_{\rm tot} \to f_r^{\rm target}$ (e.g. > 0.9) *and* $\Gamma_\eta/H \ll 1$. Freeze $\bar{\eta}$ dynamics or continue as a damped oscillator with large Γ_η.
- *BBN matching:* reach $T_{\rm reh}$ (from ρ_r) and hand off to a standard thermal history module.

11.1.4 Conservation and stability diagnostics

We monitor the following gate–aligned diagnostics each macro–step $\Delta N_{\rm macro}$:

$$\text{(Friedmann residual)} \quad \epsilon_F = \frac{|\Phi_F|}{3M_{\rm eff}^2 H^2}\,; \tag{11.2}$$

$$\text{(Energy balance)} \quad \epsilon_\rho = \frac{\left|\frac{d}{dN}\rho_{\rm tot} + 3(1 + w_{\rm eff})\rho_{\rm tot}\right|}{\rho_{\rm tot}}\,, \quad w_{\rm eff} = \frac{p_{\rm tot}}{\rho_{\rm tot}}; \tag{11.3}$$

$$\text{(Entropy production)} \quad \sigma_{\rm diss} = \frac{2\eta}{H}\, g_{\rm eff}^{ij} (\nabla_i\Phi)^*(\nabla_j\Phi) \;\ge 0 \;\;\Rightarrow\;\; \epsilon_\sigma = \max(0, -\sigma_{\rm diss}) \text{ (should be 0)}; \tag{11.4}$$

$$\text{(No–ghost gate)} \quad \lambda_{\rm min}\big(\mathsf{K}_{\rm time}\big) > 0, \;\; \epsilon_K \equiv \max\{0, -\lambda_{\rm min}\}; \tag{11.5}$$

$$\text{(Hyperbolicity bounds)} \quad 0 < c_s^2(\theta) \le 1, \;\; 0 < c_T^2 \le 1, \;\; \epsilon_c \equiv \max\{0, -c_s^2, c_s^2 - 1, -c_T^2, c_T^2 - 1\}. \tag{11.6}$$

We record $\{\epsilon_F, \epsilon_\rho, \epsilon_\sigma, \epsilon_K, \epsilon_c\}$ and enforce $\max \epsilon_i < \varepsilon_{\rm diag}$ (default $10^{-6} - 10^{-8}$) by step rejection/refinement if necessary.

11.1.5 Scalar and tensor mode integrators

We evolve Fourier modes using conformal time τ or e–fold N. Define

$$z^2(N, \theta) \equiv \frac{2a^2 M_{\rm eff}^2\, \epsilon_\eta}{c_s(\theta)}\,, \qquad v_{\mathbf{k}} \equiv z\, \zeta_{\mathbf{k}},$$

then the Mukhanov–Sasaki–like equation with anisotropic sound speed reads

$$\frac{d^2 v_{\mathbf{k}}}{d\tau^2} + \Big(c_s^2(\theta)\, k^2 - \frac{z''}{z}\Big) v_{\mathbf{k}} = 0, \qquad \text{Wronskian:} \quad v_{\mathbf{k}} v_{\mathbf{k}}^{*'} - v_{\mathbf{k}}^* v_{\mathbf{k}}' = i. \tag{11.7}$$

We integrate the *real* first–order system for $\mathrm{Re}\, v, \mathrm{Im}\, v, \mathrm{Re}\, v', \mathrm{Im}\, v'$ with an **A–stable** scheme (Radau–IIA) during horizon exit and a *phase–fitted symplectic* scheme while deeply subhorizon. Initial conditions at τ_i obey Bunch–Davies: $v_{\mathbf{k}}(\tau_i) = e^{-ikc_s\tau_i}/\sqrt{2kc_s}$, $v_{\mathbf{k}}' = -i\sqrt{kc_s/2}\, e^{-ikc_s\tau_i}$. The curvature spectrum is $\mathcal{P}_\zeta(k,\theta) = \frac{k^3}{2\pi^2} \left|\frac{v_{\mathbf{k}}}{z}\right|^2_{k=aH/c_s}$.

Tensor modes. With $u_{\mathbf{k}} = \frac{1}{2} a M_{\mathrm{eff}} h_{\mathbf{k}}$,

$$\frac{d^2 u_{\mathbf{k}}}{d\tau^2} + \Big(c_T^2 k^2 - \frac{a''}{a}\Big) u_{\mathbf{k}} = 0, \qquad u u^{*'} - u^* u' = i, \tag{11.8}$$

and $\mathcal{P}_T(k) = \frac{2k^3}{\pi^2} \left|\frac{u_{\mathbf{k}}}{aM_{\mathrm{eff}}}\right|^2_{k=aH/c_T}$. We impose $|\delta c_T| \ll 1$ during inflation and enforce the Wronskian to machine precision via *symplectic correctors* or periodic *orthonormalization*.

Numerical stability for mode evolution. We adopt:

- **Scaled variables**: integrate in N with $d\tau/dN = 1/(aH)$ to avoid catastrophic cancellations in z''/z and a''/a.
- **Embedded error control**: complex local error norms with absolute $= 10^{-12}$ and relative $= 10^{-8}$ tolerances, tightened near horizon crossing.
- **Horizon–crossing trigger**: detect $k = aH/c_s$ (or aH/c_T) and switch to superhorizon asymptotics to freeze ζ (or h).

11.1.6 Sensitivities and differentiable outputs

For inference (Chap. 10) we require gradients $\partial\mathcal{P}_\zeta/\partial\theta_i$ with respect to spectral parameters $\theta_i \in \Theta_{\mathrm{spec}}$. We implement:

- **Forward variational equations**: augment the background ODE with $\partial\mathbf{y}/\partial\theta_i$ satisfying $\frac{d}{dN}(\partial_\theta \mathbf{y}) = J\,(\partial_\theta \mathbf{y}) + \partial_\theta \mathbf{f}$, projected to the DAE manifold.
- **Continuous adjoint**: define adjoint $\lambda(N)$ solving $-\frac{d\lambda}{dN} = J^T\lambda + \partial_{\mathbf{y}}\mathcal{J}$, for objective $\mathcal{J} = \sum_{k,\theta} w_{k,\theta} \ln \mathcal{P}_\zeta$, and compute $\partial_\theta \mathcal{J} = \int \lambda^T \partial_\theta \mathbf{f}\, dN$.

Both approaches are implemented with *checkpointing* to manage memory and retain determinism.

11.1.7 Algorithmic summary (background + modes)

1. **Initialization**: choose S_0 and Θ_{spec}; set $\mathbf{y}(N_0)$ consistent with $\Phi_F = 0$; pick tolerances $(\varepsilon_{\mathrm{tol}}, \varepsilon_{\mathrm{Ftol}})$.
2. **DAE time step**: evaluate stiffness indicator $\mathcal{S}$ and select (IMEX/)Radau–IIA or BDF; advance $\mathbf{y} \to \mathbf{y}^+$ with Newton iterations; project to $\Phi_F = 0$.
3. **Diagnostics**: compute $\epsilon_F, \epsilon_\rho, \epsilon_\sigma, \epsilon_K, \epsilon_c$; if any $> \varepsilon_{\mathrm{diag}}$, reduce step and retry.
4. **Event handling**: bracket and localize end of inflation and reheating milestones; switch integration profile and tolerances.
5. **Mode evolution**: for each k–bin and viewing angle θ, integrate (11.7) and (11.8) with BD initial conditions; enforce Wronskians; record $\mathcal{P}_\zeta, \mathcal{P}_T$.
6. **Sensitivities (optional)**: run forward variational or adjoint pass to obtain $\partial\mathcal{P}/\partial\theta$.
7. **Outputs**: deliver $\{H(N), \rho_i(N)\}$, $\mathcal{P}_\zeta(k,\theta)$, $\mathcal{P}_T(k)$, and gate logs for downstream emulation and inference.

Reproducibility. All runs store: (i) code hash and solver settings; (ii) tolerance and gate thresholds; (iii) random seeds for any stochastic components (none used here); and (iv) a step–by–step audit of ϵ–diagnostics. Default settings target double–precision reproducibility across platforms.

11.2 Spectral Lattice / Mesh for Resonance and Defects

Scope. We describe a high–fidelity pseudo–spectral framework to simulate (i) the *resonant interference* of unstable modes seeded during the η–driven epoch and (ii) the *formation, tracking, and interactions* of topological/solitonic defects in the structural sector $S = \{\rho, \eta, \Phi, \chi^\mu, T\}$. The scheme couples Fourier differentiation (spectral accuracy for smooth fields) to *absorbing layers* (PML) for outgoing radiation and to *topology–aware* diagnostics (winding, monopole charge, Hopf invariant). All numerics obey the Volume I gate philosophy (stability, causality, restoration, topology, numerics).

11.2.1 Computational domain, grids, and transforms

We consider a rectangular box $\Omega = [0, L_x) \times [0, L_y) \times [0, L_z)$ with uniform mesh

$$\mathbf{x_n} = (n_x\Delta x,\ n_y\Delta y,\ n_z\Delta z), \quad \Delta x = L_x/N_x,\ \Delta y = L_y/N_y,\ \Delta z = L_z/N_z,$$

and spectral wavenumbers $k_x = \frac{2\pi}{L_x} m_x$ with $m_x \in \{-\lfloor N_x/2\rfloor, \dots, \lfloor (N_x-1)/2\rfloor\}$ (and analogously for y, z). FFTs map fields $f(\mathbf{x})$ to $\hat{f}(\mathbf{k})$. Spatial derivatives are applied as $\partial_i f \leftrightarrow ik_i\hat{f}$. Anisotropy from the director $\hat{\chi}$ and the effective metric $g^{\rm eff}_{ij}$ is implemented via the quadratic form

$$k_g^2 \ \equiv\ g^{ij}_{\rm eff} k_i k_j \qquad \Rightarrow \qquad \Delta_{g_{\rm eff}} f\,\widehat{\ } \ =\ -\,k_g^2\,\hat{f}.$$

For time stepping we adopt a uniform Δt (physics time) or ΔN (e–folds) with adaptive control near events (Sec. 11.1.3).

Dealiasing and filtered operators. To control spectral blocking in nonlinearity, we employ the 2/3–rule: zero all modes with $|m_i| > \frac{1}{3}N_i$. A smooth exponential filter $\sigma_p(\mathbf{k}) = \exp\big[-\alpha(k/k_{\max})^{2p}\big]$ ($p\sim 8$, $\alpha\sim 36$) is optionally applied to non–primary fields to damp grid–scale noise without biasing conserved quantities.

11.2.2 Equations advanced on the lattice

We evolve the structural/matter fields in comoving aether gauge ($\delta T = 0$) on a fixed background $a(t)$, $H(t)$ supplied by Sec. 11.1. A representative real–space system (schematic) reads

$$\partial_t\rho \ =\ \mathcal{D}_\rho[\rho, \Phi, \chi;\, g_{\rm eff}] \ -\ \Gamma_\rho\,\Pi_\rho\,, \tag{11.9a}$$

$$\kappa_\eta(\mathcal{S})\big(\partial_t^2\eta + 3H\partial_t\eta\big) - \kappa_\eta\,\Delta_{g_{\rm eff}}\eta + V'_\eta(\eta) \ =\ -\,\Gamma_\eta(\mathcal{S})\,\partial_t\eta \ +\ \mathcal{J}_\eta[\Phi, S], \tag{11.9b}$$

$$\mathcal{K}_\Phi(\mathcal{S})\big(\partial_t^2\Phi + 3H\partial_t\Phi\big) - \mathcal{Z}^{ij}_\Phi(\mathcal{S})\,\nabla_i\nabla_j\Phi + \frac{\partial U}{\partial\Phi^\dagger} \ =\ -\,\Gamma_\Phi\,\partial_t\Phi \ +\ \mathcal{J}_\Phi[\eta, S], \tag{11.9c}$$

$$\mathcal{M}_\mu{}^\nu(\mathcal{S})\big(\partial_t^2\chi^\mu + 3H\partial_t\chi^\mu\big) - \mathcal{Z}^{ij}_\mu(\mathcal{S})\,\nabla_i\nabla_j\chi^\nu + \frac{\partial W}{\partial\chi_\nu} \ =\ -\,\Gamma_\chi\,\partial_t\chi^\nu \ +\ \mathcal{J}_\chi[\eta, \Phi, S], \tag{11.9d}$$

with $\mathcal{D}_\rho$ a continuity–like operator (including spectral diffusion), and $\mathcal{K}_\Phi, \mathcal{Z}_\Phi, \mathcal{M}, \mathcal{Z}_\mu$ the kinetic/-gradient tensors from the unified action (Vol. I, Ch. 3). The emergent curvature sector enters via $g^{\rm eff}_{ij}(S)$ (we work in the Cowling–like approximation where its backreaction on background is handled by $\{H, M_{\rm eff}\}$). Dissipation coefficients $\Gamma_{\eta,\Phi,\chi} \geq 0$ enforce the entropy gate.

Hamiltonian density and monitoring. The lattice Hamiltonian (coarse–grained) is

$$\begin{aligned}\mathcal{H} = {}& \tfrac{1}{2}\kappa_\eta\big(|\partial_t\eta|^2 + |\nabla\eta|_g^2\big) + V_\eta(\eta) + \tfrac{1}{2}\mathcal{K}_\Phi|\partial_t\Phi|^2 + \tfrac{1}{2}\mathcal{Z}^{ij}_\Phi\partial_i\Phi^\dagger\partial_j\Phi + U(\Phi)\\ &+ \tfrac{1}{2}\mathcal{M}_{\mu\nu}\partial_t\chi^\mu\partial_t\chi^\nu + \tfrac{1}{2}\mathcal{Z}^{ij}_{\mu\nu}\partial_i\chi^\mu\partial_j\chi^\nu + W(\chi) + \mathcal{H}_{\rm int}[\Phi, S],\end{aligned} \tag{11.10}$$

with $|\nabla\eta|_g^2 \equiv g^{ij}_{\rm eff}\partial_i\eta\,\partial_j\eta$. We track $E(t) = \int_\Omega \mathcal{H}\,d^3x$ and the dissipative budget $\dot{E}_{\rm diss} = -\int(\Gamma_\eta|\partial_t\eta|^2 + \Gamma_\Phi|\partial_t\Phi|^2 + \Gamma_\chi|\partial_t\chi|^2)\,d^3x$.

11.2.3 Time integration: ETD/Strang–IMEX and CFL

We split $\partial_t \mathbf{u} = \mathcal{L}\mathbf{u} + \mathcal{N}(\mathbf{u}) - \mathcal{D}(\mathbf{u})$, where $\mathcal{L}$ are linear dispersive terms (diagonal in Fourier space), $\mathcal{N}$ nonlinear local terms, and $\mathcal{D}$ Rayleigh–type dissipation. Two families are provided:

- **ETDRK4 (exponential time differencing)** for mildly stiff wave–like regimes. In Fourier space,

$$\hat{\mathbf{u}}_{n+1} = e^{\Delta t \mathsf{L}} \hat{\mathbf{u}}_n + \sum_{j=1}^{4} \varphi_j(\Delta t \mathsf{L})\, \widehat{\mathcal{R}_j},$$

  with L the symbol of $\mathcal{L}$, φ_j the ETD φ–functions (computed via scaling–squaring/Padé), and $\mathcal{R}_j$ RK residuals of $\mathcal{N} - \mathcal{D}$.

- **Strang–IMEX**: half–step implicit on $\mathcal{D}$ (and stiff parts of $\mathcal{L}$ if needed), full explicit step on $\mathcal{N}$, then half–step implicit. The implicit solve is diagonal in Fourier space for linear $\mathcal{D}$.

A spectral CFL controller limits Δt by the largest phase speed and damping:

$$\Delta t \le \min_{\mathbf{k}} \left\{ \frac{\mathrm{CFL}}{\sqrt{c_{\max}^2 k_g^2 + m_{\mathrm{eff}}^2}},\ \frac{1}{\gamma_{\max}} \right\},$$

with $c_{\max} = \max\{c_{\parallel,\eta}, c_{\perp,\eta}, \ldots\}$ and $\gamma_{\max} = \max\{\Gamma_\eta, \Gamma_\Phi, \Gamma_\chi\}$.

11.2.4 Absorbing boundaries: perfectly matched layers (PML)

To suppress spurious reflections, we envelop Ω by PML slabs of thickness d_{PML} in each direction. For a generic field u, define damping profiles $\sigma_i(x_i)$ that rise smoothly from 0 inside to $\sigma_{i,\max}$ at the outer boundary, e.g.

$$\sigma_i(\xi) = \sigma_{i,\max} \left(\frac{\xi}{d_{\mathrm{PML}}} \right)^p, \qquad \xi \in [0, d_{\mathrm{PML}}], \quad p \ge 2.$$

The wave equation $\partial_t^2 u - c^2 \Delta u = 0$ is split (auxiliary variables ψ_i) into

$$\partial_t^2 u - c^2 \sum_i \partial_i(\partial_i u + \psi_i) = 0, \qquad \partial_t \psi_i + \sigma_i \psi_i = \sigma_i \partial_i u, \tag{11.11}$$

and analogously for multi–field systems; in Fourier space, PML enters as $\partial_i \to \partial_i + \mathcal{S}_i$ with $\mathcal{S}_i = \sigma_i * \partial_t^{-1}$. The parameters $(\sigma_{i,\max}, p, d_{\mathrm{PML}})$ are tuned to keep reflections $< 10^{-6}$ at target frequencies.

11.2.5 Defect finders and topological diagnostics

Vortices/strings (π_1). For a complex order parameter $\Xi = |\Xi| e^{i\alpha}$ (constructed from Φ or a composite), the winding number through an elementary plaquette P is

$$n_P = \frac{1}{2\pi} \operatorname{wrap}\left(\sum_{(i \to j) \in P} \Delta\alpha_{ij} \right), \qquad \Delta\alpha_{ij} = \operatorname{atan2}\big(\operatorname{Im}(\Xi_j \Xi_i^*),\ \operatorname{Re}(\Xi_j \Xi_i^*) \big),$$

with wrap mapping to $(-\pi, \pi]$. A nonzero n_P flags a string core piercing P. Segment–linking (union–find) reconstructs string world–sheets; tension $\mathcal{T}$ is extracted from the localized energy per unit length.

Domain walls (π_0). For a real order parameter Θ distinguishing vacua $v_\pm$, walls are iso–surfaces $\Theta = \Theta_\star$ separating regions; marching cubes yields triangulated surfaces. The tension σ follows from surface integrals of $\mathcal{H}$.

Monopoles (π_2). For a normalized vector $\mathbf{n} : \mathbb{R}^3 \to S^2$, compute lattice magnetic flux through the surface of a cube C surrounding a node:

$$Q_C = \frac{1}{4\pi} \sum_{F \in \partial C} \Omega_F, \qquad \Omega_F = \operatorname{atan2}\big(\mathbf{n}_a \cdot (\mathbf{n}_b \times \mathbf{n}_c),\, 1 + \mathbf{n}_a \cdot \mathbf{n}_b + \mathbf{n}_b \cdot \mathbf{n}_c + \mathbf{n}_c \cdot \mathbf{n}_a\big),$$

with Ω_F the spherical excess on face F (triangulated), robust to mesh rotation.

Hopfions/Skyrmions (π_3). Given $\mathbf{n}(\mathbf{x}) \in S^2$, define a lattice gauge potential $\mathbf{A}$ by solving $\nabla \times \mathbf{A} = \mathbf{F}$, where $F_{ij} = \mathbf{n} \cdot (\partial_i \mathbf{n} \times \partial_j \mathbf{n})$. The discrete Hopf invariant is

$$Q_H = \frac{1}{32\pi^2} \sum_{\text{cells}} \epsilon^{ijk} A_i F_{jk}\, \Delta V,$$

constructed with centered differences and FFT–Poisson solve for $\mathbf{A}$ in Coulomb gauge. Topology gate: Q_H integer to within $< 10^{-2}$ under refinement.

11.2.6 Resonance constructor and Chladni fields

To initialize or analyze resonance patterns (Chap. 7), we synthesize a field

$$\mathcal{I}(\mathbf{x}) = \sum_{n=1}^{N_{\text{mode}}} A_n \cos\big(\mathbf{k}_n \cdot \mathbf{x} + \varphi_n\big), \quad \mathbf{k}_n \in \arg\max_{\mathbf{k}} \lambda_{\max}(\mathbf{k}),$$

with $\lambda_{\max}(\mathbf{k})$ from the linear dispersion (Sec. **??**). Amplitudes/phases are drawn from controlled ensembles to match the instability band and anisotropy relative to $\hat{\chi}$. The nodal set of $\mathcal{I}$ predicts ridge/void skeletons; we quantify morphology by Minkowski functionals and skeleton statistics (Chap. 7).

11.2.7 Validation, convergence, and gate logs

Linear validation. Evolve small–amplitude waves and compare numerically measured dispersion $\omega(\mathbf{k})$ to analytic $\omega^2 = c_\parallel^2 k_\parallel^2 + c_\perp^2 k_\perp^2 + m^2$. Require relative error $< 10^{-4}$ up to $k \sim 0.6\, k_{\text{Ny}}$.

Topological stability. Track (n_P, Q_C, Q_H) under (i) mesh refinement $N \to 2N$, (ii) box scaling at fixed physical resolution, and (iii) moderate filter strength. Quantized invariants must be mesh–invariant within 10^{-2}.

Energy and entropy budgets. Verify

$$\frac{dE}{dt} \simeq -\int_\Omega \left(\Gamma_\eta |\partial_t \eta|^2 + \Gamma_\Phi |\partial_t \Phi|^2 + \Gamma_\chi |\partial_t \chi|^2\right) d^3x \;-\; \mathcal{F}_{\text{PML}},$$

with $\mathcal{F}_{\text{PML}}$ the energy flux into PML, and non–negative discrete entropy production. Gate tolerances mirror Sec. 11.1.4.

11.2.8 Parallelization, I/O, and reproducibility

We employ pencil–decomposed FFTs (MPI) with slab fallback; domain decomposition aligns PML buffers per rank. Checkpoint–restart writes $\{\rho, \eta, \Phi, \chi^\mu\}$ plus solver state every Δt_{chk}. HDF5 files store datasets with attributes: code commit, compile flags, gate tolerances, filter/dealias settings, RNG seeds for resonance synthesis, and a JSON gate log (time series of diagnostics). All post–processing (defect catalogs, morphology) reads from these checkpoints to ensure end–to–end reproducibility.

11.2.9 Algorithmic summary (lattice)

1. **Setup:** choose box Ω, resolution (N_x, N_y, N_z), and PML parameters; build wavenumber arrays and metric–aware symbols k_g^2.

2. **Initialize:** import $a(t), H(t)$ and $M_{\rm eff}(t)$; seed fields by linear instability eigenmodes or synthetic $\mathcal{I}(\mathbf{x})$; zero defect maps.

3. **Advance:** apply ETDRK4 or Strang–IMEX with dealiasing/filter; enforce CFL; update PML auxiliaries.

4. **Diagnostics:** every $n_{\rm diag}$ steps compute $\mathcal{H}$, budgets, gate errors, and run defect finders; write logs and light–weight snapshots.

5. **Refinement (optional):** perform p–refinement (increase N_i) or h–refinement (local AMR proxy via windowed FFT) around defect cores; reinterpolate with spectral accuracy.

6. **Outputs:** defect catalogs (strings, walls, monopoles, Hopfions), morphology statistics, and energy/entropy histories for comparison with analytic predictions (Chaps. 7, 5).

11.3 Synthetic Maps and Pipeline Validation

Scope. We provide an end–to–end (E2E) framework to generate *synthetic skies* from spectral–cosmology parameters, propagate them through survey–like systematics, and validate inference pipelines against known truths. Deliverables are: (i) mock CMB/LSS/lensing/GW maps and catalogs; (ii) reference two– and three–point statistics with covariances; (iii) recovery tests for $(n_s, r, f_{\rm NL}^{\rm eff}, g_*, c_T, \dots)$; (iv) gate logs certifying stability, causality, restoration, topology, and numerics (S/G/TQ/D/N).

11.3.1 Forward–model blueprint (parameters → skies)

Given a spectral parameter vector

$$\mathbf{\Theta}_{\rm spec} = \{V_\eta(\cdot), \kappa_\eta(\cdot), \Gamma_\eta(\cdot), M_{\rm eff}(\rho_0),\ c_s(\theta),\ c_T,\ \text{defect sector}\},$$

the forward model proceeds as:

1. **Background & primordial:** Integrate background $a(t)$ and slow parameters ϵ_η, η_η (Sec. **??**); compute primordial spectra $\mathcal{P}_\zeta(k,\theta),\ \mathcal{P}_T(k)$ with anisotropy kernel $\mathcal{A}(\hat{\mathbf{k}}\cdot\hat{\chi})$ (§4.1–4.2).

2. **Transfer to late time:** Evolve with a Boltzmann kernel $\mathsf{T}_X(k, z; \mathbf{\Theta}_{\rm spec})$ to obtain late–time fields $X \in \{\delta,\ \phi_{\rm CMB},\ \kappa,\ v,\ \gamma\}$.

3. **Line–of–sight projections:** Build observables with survey windows $W(\chi)$:

$$\Delta_{\rm g}(\hat{\mathbf{n}}) = \int_0^{\chi_{\rm max}} d\chi\ W_{\rm g}(\chi)\, b(\chi)\, \delta(\chi\hat{\mathbf{n}}, \chi) + \epsilon_{\rm shot}, \tag{11.12}$$

$$\kappa(\hat{\mathbf{n}}) = \frac{3H_0^2\Omega_m}{2} \int_0^{\chi_s} d\chi\ \frac{\chi(\chi_s - \chi)}{\chi_s a(\chi)}\, \delta(\chi\hat{\mathbf{n}}, \chi) + \epsilon_\kappa, \tag{11.13}$$

$$T(\hat{\mathbf{n}}) = \int d\chi\ \mathcal{S}_T[\Phi, \Psi; \mathbf{\Theta}_{\rm spec}] + \epsilon_T, \qquad E, B \text{ analogous}, \tag{11.14}$$

$$h_A(f, \hat{\mathbf{n}}) = \int d\chi\ \mathcal{S}_h^{(A)}(f, \chi; \mathbf{\Theta}_{\rm spec}) + n_A,\ A \in \{+, \times\}, \tag{11.15}$$

   including η–induced modifications of sources $\mathcal{S}$ and, for GWs, a possible $c_T \neq 1$.

4. **Survey layers:** Apply beam/PSF B_ℓ, transfer F_ℓ, masks $M(\hat{\mathbf{n}})$, selection functions, noise realizations (white + $1/f$ for CMB; Poisson + systematics templates for LSS).

5. **Statistics:** Estimate $\widehat{C}_\ell$ (pseudo–C_ℓ) and bispectrum $\widehat{B}_{\ell_1\ell_2\ell_3}$ (binned, separable).

11.3.2 Pseudo-C_ℓ and covariance with mask and noise

Let $X, Y \in \{T, E, B, \Delta_{\rm g}, \kappa\}$ and masked fields $\widetilde{X} = MX$, $M(\hat{\mathbf{n}}) \in [0,1]$. Spherical–harmonic coefficients $\tilde{a}^X_{\ell m}$ are coupled by the mask with mode–coupling matrix $\mathsf{M}_{\ell\ell'}$:

$$\langle \tilde{C}^{XY}_\ell \rangle \equiv \frac{1}{2\ell+1}\sum_m \tilde{a}^X_{\ell m}\tilde{a}^{Y*}_{\ell m} = \sum_{\ell'} \mathsf{M}_{\ell\ell'}\, C^{XY}_{\ell'} + N^{XY}_\ell, \qquad \mathsf{M}_{\ell\ell'} = \frac{2\ell'+1}{4\pi}\sum_L (2L{+}1)\, W_L \begin{pmatrix} \ell & \ell' & L \\ 0 & 0 & 0 \end{pmatrix}^2, \tag{11.16}$$

where W_L is the mask power spectrum and the $3j$ symbol encodes rotational coupling. A MASTER–like deconvolution gives an unbiased estimator $\widehat{C}_\ell = \sum_{\ell'} \mathsf{M}^{-1}_{\ell\ell'}\big(\tilde{C}_{\ell'} - N_{\ell'}\big)$. Gaussian covariance (with anisotropic correction) reads

$$\mathrm{Cov}\Big[\widehat{C}^{XY}_\ell, \widehat{C}^{WZ}_{\ell'}\Big] \simeq \frac{\delta_{\ell\ell'}}{(2\ell+1)\Delta\ell\, f^{\rm eff}_{\rm sky}} \Big(\widehat{C}^{XW}_\ell \widehat{C}^{YZ}_\ell + \widehat{C}^{XZ}_\ell \widehat{C}^{YW}_\ell\Big) + \mathcal{O}(\mathrm{NG}), \tag{11.17}$$

with $f^{\rm eff}_{\rm sky}$ the effective sky fraction and $\Delta\ell$ the bin width.

11.3.3 Estimators for anisotropy and non–Gaussianity

Statistical anisotropy (g_*). Model a quadrupolar modulation of the primordial curvature: $\mathcal{P}_\zeta(\mathbf{k}) = \mathcal{P}_0(k)\big[1 + g_*\,(\hat{\mathbf{k}}\cdot\hat{\chi})^2\big]$. A quadratic maximum–likelihood (QML) estimator in harmonic space is

$$\widehat{g}_* = \frac{1}{\mathcal{N}}\sum_{\ell m}\sum_{\ell' m'} a^X_{\ell m}\, \Xi^{\ell\ell'}_{mm'}\, a^{X*}_{\ell' m'}, \qquad \Xi^{\ell\ell'}_{mm'} = \sum_M \alpha_{\ell\ell'}\, \mathcal{G}^{2M}_{\ell m, \ell' m'}\, Y_{2M}(\hat{\chi}), \tag{11.18}$$

with $\mathcal{G}$ Gaunt integrals and $\alpha_{\ell\ell'}$ a response kernel precomputed from transfer functions. The normalization $\mathcal{N}$ and Fisher error $\sigma(g_*)$ follow from Monte–Carlo with $\hat{\chi}$ randomized on the sphere.

Bispectrum and $f^{\rm eff}_{\rm NL}$. For a separable template $S(\ell_1, \ell_2, \ell_3)$ (e.g. local/equilateral with angular dependence inherited from $c_s(\theta)$), the KSW–type estimator is

$$\widehat{f}_{\rm NL} = \frac{1}{\mathcal{N}_B}\sum_{\{\ell_i m_i\}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \frac{a_{\ell_1 m_1} a_{\ell_2 m_2} a_{\ell_3 m_3}}{C_{\ell_1} C_{\ell_2} C_{\ell_3}}\, S(\ell_1, \ell_2, \ell_3), \qquad \mathcal{N}_B \text{ from } \langle \widehat{f}^2_{\rm NL}\rangle. \tag{11.19}$$

Anisotropic shapes enter by replacing $S \to S(\ell_i)\,\mathcal{Y}_{LM}$ with low–L modulations and including linear–term subtractions for mask/noise.

11.3.4 Mock catalogs and map–making

CMB. Draw Gaussian $a^X_{\ell m}$ with input $\mathcal{P}_\zeta, \mathcal{P}_T$ (and optional weak NG via modal methods), filter by $B_\ell F_\ell$, transform to HEALPix maps at $N_{\rm side}$, add anisotropic noise $\mathcal{N}(\hat{\mathbf{n}})$, and apply masks M_T, M_P.

LSS (galaxies). Simulate a lognormal density $\delta_{\rm LN}$ matching target $C^{\Delta\Delta}_\ell$; Poisson sample galaxies with selection $n(z)$ and depth variations; imprint bias $b(z)$ and redshift errors via a transfer operator in harmonic space.

Weak lensing. Generate convergence κ slices and shear (γ_1, γ_2); add shape noise with dispersion $\sigma_e(z)$ and multiplicative bias $m(\hat{\mathbf{n}})$ templates; build tomographic bins.

GW standard sirens / PTA. For a given c_T, generate arrival–time residuals or strain h_A with sky distribution consistent with catalogs; inject dispersion–like effects if allowed by the curvature sector; add detector noise.

11.3.5 E2E validation: recovery, coverage, and null tests

Parameter recovery. From mocks, run the full inference pipeline (Chap. 10) to obtain posteriors $p(\mathbf{\Theta}_{\rm spec}|\text{data})$. Check unbiasedness and *coverage* by repeating over $N_{\rm sim}$ realizations: the nominal 68% credible region should contain the truth in $0.68 \pm \mathcal{O}(N^{-1/2}_{\rm sim})$ of cases.

Consistency gates on mocks.

- **G1 (restoration):** As $g_* \to 0,\ c_T \to 1$, recovered spectra agree with ΛCDM baselines within cosmic–variance.
- **G2 (causality):** No superluminal fits for $c_s(\theta), c_T$ within prior support; likelihood penalizes unphysical regions.
- **G3 (soft limits):** Squeezed–limit bispectrum of mocks satisfies spectral soft theorems (cf. Chap. 4); verify by template projections.
- **G4 (topology):** If defect injections are present, recovered string/wall statistics match inputs within UQ bands (Chap. 5).
- **N (numerics):** Map–level power and bispectra converge under $N_{\rm side}$ and bandpower binning refinements.

Null tests (per–survey). Cross–spectra that must vanish by symmetry (e.g. TB, EB for parity–even primordial sector) are consistent with zero; curl lensing B-mode maps show no spurious power beyond simulations; spatial splits (hemisphere, depth, time) pass χ^2 consistency; *blinding* is enforced via hidden nuisance scalings until all null tests pass.

11.3.6 Uncertainty quantification (UQ) and reproducibility

We propagate uncertainties via:

- **Monte Carlo ensembles** ($N_{\rm MC} \sim 10^2$–10^3) with distinct seeds to estimate sampling covariances and non–Gaussian errors.
- **Jackknife/Bootstrap** on sky partitions for robustness to spatially varying systematics.
- **Nuisance marginalization** with priors informed by calibration pipelines; hyperparameters stored alongside mocks.

All mocks and intermediate products are written to HDF5 with metadata: code commit hash, YAML config, RNG seeds, beam/mask hashes, and gate–log JSON (thresholds and pass/fail flags). A single `.replay` file reconstitutes the entire E2E run.

11.3.7 Algorithmic summary (E2E pipeline)

1. **Choose $\boldsymbol{\Theta}_{\rm spec}$** and survey configs; compute primordial spectra with anisotropy/NG options.
2. **Propagate** to late time with transfer functions; assemble projected observables using Eqs. (11.12)–(11.13).
3. **Instrumentalize** with beams, masks, and noise; generate mocks (maps, catalogs).
4. **Analyze** with pseudo–C_ℓ, bispectrum estimators, and likelihood samplers; obtain posteriors.
5. **Validate** recovery, coverage, and null tests; record gate–log; iterate if any gate fails.

Summary. This section establishes a reproducible E2E machinery linking spectral–cosmology inputs to survey–level outputs. It quantifies biases and uncertainties, enforces physical gates, and ensures that any claimed signature (anisotropy, NG, defects, $c_T \neq 1$) survives realistic analysis conditions.

Part V

Consistency Gates, ΛCDM Comparison, and Predictions

Chapter 12

Consistency Gates, ΛCDM Comparison, and Predictions

12.1 Global Gates (S/G/TQ/D) and Passing Criteria

Scope. We define a *global* consistency layer that carves the viable domain of spectral cosmology in parameter space, enforces first–principles constraints, and interfaces cleanly with data–driven likelihoods. Let

$$\mathbf{\Theta}_{\rm spec} \equiv \Big\{ V_\eta(\cdot),\ \kappa_\eta(\cdot),\ \Gamma_\eta(\cdot),\ M_{\rm eff}(\rho_0),\ c_s(\theta),\ c_T,\ g_*,\ f_{\rm NL}^{\rm eff},\ \text{defect sector } \mathcal{D},\ \text{curvature couplings, reheating map } \mathcal{R} \Big\}$$

collect the background, perturbative, and defect–sector parameters (defined in preceding chapters). The *allowed region* is the set

$$\mathcal{A} = \Big\{ \mathbf{\Theta}_{\rm spec} \,\Big|\, \underbrace{\mathsf{S}(\mathbf{\Theta})}_{\text{Stability}} = \underbrace{\mathsf{G}(\mathbf{\Theta})}_{\text{Geometry/Causality}} = \underbrace{\mathsf{TQ}(\mathbf{\Theta})}_{\text{Topology/Quantum}} = \underbrace{\mathsf{D}(\mathbf{\Theta})}_{\text{Data/Phenomenology}} = 1 \Big\}. \tag{12.1}$$

We now specify each gate as a set of *sharp inequalities/equalities* and *structured priors* suitable for sampling.

Gate S: Dynamical stability and well–posedness

- **No ghosts (time–kinetic positivity).** For the quadratic action of all propagating sectors (scalars, vectors, tensors) about the relevant background,

$$\mathsf{K}(\mathbf{\Theta}) \succ 0, \qquad \text{i.e.} \quad \forall \mathbf{v} \neq 0: \ \mathbf{v}^\top \mathsf{K}\, \mathbf{v} > 0. \tag{12.2}$$

- **Gradient stability and hyperbolicity.** The principal symbol in the aether frame u^μ, $\mathsf{P}(\hat{\mathbf{k}}) = \mathsf{Z}^{ij}\hat{k}_i\hat{k}_j$, has nonnegative eigenvalues and defines a causal cone:

$$0 \le c_\perp^2(\Theta) \le c_\parallel^2(\Theta) \le 1, \qquad \det\big[\lambda^2\mathsf{K} + \mathsf{Z}^{ij}k_ik_j\big] \text{ hyperbolic } \forall \hat{\mathbf{k}}. \tag{12.3}$$

- **Dissipative consistency.** Spectral Rayleigh/GKSL coefficients are positive:

$$\Gamma_\eta(\mathcal{S}) \ge 0, \qquad \text{GKSL generator } \mathcal{L} \text{ completely positive and trace preserving (CPTP).} \tag{12.4}$$

- **Inflationary slow–roll window (when invoked).** $\epsilon_\eta \ll 1$, $|\eta_\eta| \ll 1$ and $\dot{M}_{\rm eff}/(HM_{\rm eff}) \ll 1$ over $N \gtrsim 50$ e–folds.

Gate G: Geometric/causal consistency under the emergent metric

- **Lorentzian signature and aether frame.** $g_{\mu\nu}^{\rm eff}(\mathbf{\Theta})$ has signature $(-,+,+,+)$ and admits a unit timelike u^μ.

- **Tensor speed bound (multi–messenger).**

$$\left|c_T - 1\right| \le \varepsilon_T^{\rm data}, \qquad \varepsilon_T^{\rm data} \text{ (tight; e.g. } \lesssim 10^{-15} \text{ on cosmic times for nearby events).} \tag{12.5}$$

- **Statistical anisotropy bound during inflation/primordial.**

$$\left|g_*\right| \le \varepsilon_{\rm aniso}^{\rm data}, \qquad \frac{|c_\parallel - c_\perp|}{\frac{1}{2}(c_\parallel + c_\perp)} \le \varepsilon_{c_s}^{\rm data}. \tag{12.6}$$

- **Restoration gate (late time).** As the η–coherence decays,

$$c_T \to 1, \quad c_s(\theta) \to c_s^{\rm iso}, \quad \text{effective Einstein equations reduce to GR–like form with } M_{\rm eff} \to \text{const.} \tag{12.7}$$

Gate TQ: Topological and quantum consistency

- **Soft theorems and consistency relations.** The squeezed limit of the bispectrum and soft graviton limits obey the spectral single–clock relations:

$$\lim_{k_L \to 0} B_\zeta(\mathbf{k}_L, \mathbf{k}_S, -\mathbf{k}_S - \mathbf{k}_L) \;=\; -\,(n_s - 1)\; P_\zeta(k_L) P_\zeta(k_S) + \mathcal{O}(g_*), \tag{12.8}$$

  with controlled anisotropic corrections $\propto g_*$.

- **Ward identities (gauge/isometry).** Emergent gauge–current conservation and diffeo–Ward identities hold (up to controlled dissipative terms) in correlators and transfer kernels.

- **Topological charge quantization and inflow.** Defect charges (strings, walls, Hopfions) satisfy integer quantization and, where applicable, spectral WZW/CS inflow cancels boundary variations (no global anomalies).

- **Unitary/inclusive IR finiteness.** Inclusive rates respect KLN–type cancellations (soft spectral/gravitational emissions), matching the open–system width inferred from GKSL (§11.3).

Gate D: Data/phenomenology

- **Baseline observables.** Angle–averaged primordial spectrum and tensors within bounds:

$$n_s(\mathbf{\Theta}) \in \mathcal{I}_{n_s}^{\rm data}, \quad r(\mathbf{\Theta}) \le r_{\rm max}^{\rm data}, \quad f_{\rm NL}^{\rm eff}(\text{shape}) \in \mathcal{I}_{\rm NG}^{\rm data}. \tag{12.9}$$

- **Late–time fits.** LSS/lensing/SNe/BAO/GW constraints on $w_{\rm eff}(z) = -1 + \delta(z)$, $H(z)$, $P(k)$, $C_\ell^{\kappa\kappa}$, etc. implemented by the composite likelihood

$$-2\ln \mathcal{L}_{\rm data}(\mathbf{\Theta}) = \sum_{d \in \mathcal{D}_{\rm sets}} \big(\mathbf{d} - \mathbf{m}_d(\mathbf{\Theta})\big)^\top \mathbf{C}_d^{-1} \big(\mathbf{d} - \mathbf{m}_d(\mathbf{\Theta})\big) + \text{penalties for systematics.} \tag{12.10}$$

- **Defect sector bounds.** If defects are active, $G\mu$, $n_{\rm wall}$, Hopf density respect CMB and lensing limits; SIDM–like self–interaction per mass satisfies $(\sigma/M)_{\rm sol} \in \mathcal{I}_{\sigma/M}^{\rm astro}$.

Passing criteria and sampling implementation

Hard/soft enforcement. We encode S/G/TQ as *hard* constraints (⇒ prior probability zero if violated) and D as a *soft* constraint via the likelihood:

$$\Pi(\mathbf{\Theta}) \;=\; \underbrace{\mathbf{1}_{\mathsf{S}=1}\, \mathbf{1}_{\mathsf{G}=1}\, \mathbf{1}_{\mathsf{TQ}=1}}_{\text{hard gates}} \times \exp\Big(-\tfrac{1}{2}\,\chi^2_{\rm data}(\mathbf{\Theta})\Big), \qquad \chi^2_{\rm data} \equiv -2\ln \mathcal{L}_{\rm data}. \tag{12.11}$$

A sample $\mathbf{\Theta}$ *passes* if $\mathsf{S} = \mathsf{G} = \mathsf{TQ} = 1$ and $\chi^2_{\rm data} \le \chi^2_{\rm thr}$ for the adopted confidence level.

Algorithmic sieve. Given a proposal $\mathbf{\Theta}$:

1. Build $g^{\rm eff}_{\mu\nu}, u^\mu$ and quadratic operators $\mathsf{K}, \mathsf{Z}^{ij}$.
2. Check S: Eqs. (12.2)–(12.4).
3. If pass, check G: Eqs. (12.5)–(12.7).
4. If pass, check TQ: Eqs. (12.8) and Ward/inflow conditions.
5. If pass, evaluate D via (12.10); accept if χ^2 below threshold.

Reporting. For any admitted point, record the gate–vector $\mathbf{g} = (\mathsf{S}, \mathsf{G}, \mathsf{TQ}, \mathsf{D})$ and diagnostic margins $\Delta c_T = \varepsilon_T^{\rm data} - |c_T - 1|$, $\Delta g_* = \varepsilon_{\rm aniso}^{\rm data} - |g_*|$, $\lambda_{\min}(\mathsf{K})$, etc. The boundary of $\mathcal{A}$ is determined by the vanishing of one or more margins.

Summary. Gate S guarantees well–posed dynamics; Gate G ties the construction to a causal emergent geometry and late–time restoration; Gate TQ enforces soft/ward/inflow consistency and topological quantization; Gate D confronts the theory with data. Together they define a mathematically crisp, operationally usable $\mathcal{A}$ in which the model is both *consistent* and *predictive*.

12.2 Where Spectral Cosmology Wins or Fails vs. ΛCDM

Purpose. Having delimited the admissible parameter region $\mathcal{A}$ by the global gates (§12.1), we compare *predictions* of spectral cosmology (SC) against the ΛCDM baseline. We organize this comparison along three axes: (i) *equivalence regime* where SC must reproduce ΛCDM within current errors; (ii) *discriminants* where SC robustly departs; (iii) *failure modes* that would falsify SC.

I. Equivalence regime (must-match domain)

SC reduces to an effectively single-clock, nearly isotropic inflationary and late-time expansion history once the η-coherence damps and the restoration gate holds:

$$c_T \to 1, \qquad c_s(\theta) \to c_s^{\rm iso}, \qquad w_{\rm eff}(z) \to -1 \text{ (up to small } \delta(z)). \tag{12.12}$$

In this domain, the following must be satisfied within present uncertainties:

$$n_s \simeq 0.96\text{–}0.97, \qquad r \lesssim 10^{-2}\text{–}10^{-1} \text{ (experiment dependent)}, \qquad f_{\rm NL}^{\rm local} \approx 0 \pm \mathcal{O}(1), \tag{12.13}$$

$$H(z) \text{ and } D_A(z) \text{ consistent with BAO/SNe}, \qquad \Delta P(k)\big|_{\rm lin}/P(k) \ll 1 \text{ on } k \lesssim 0.1\, h\,{\rm Mpc}^{-1}. \tag{12.14}$$

Any *systematic* failure here constitutes an exclusion (Gate D violation).

II. Robust discriminants (where SC should win/lose)

We list observables where SC yields *controlled, calculable* deviations from ΛCDM:

1. **Statistical anisotropy (primordial).** SC generically predicts a suppressed but nonzero quadrupolar modulation of the scalar spectrum,

$$\mathcal{P}_\zeta(\mathbf{k}) = \mathcal{P}_\zeta^{(0)}(k)\left[1 + g_*\left(\hat{\mathbf{k}}\cdot\hat{\chi}^{(0)}\right)^2 + \cdots\right], \tag{12.15}$$

with $|g_*| \ll 1$ set by $|c_\parallel - c_\perp|$ during the η-wave stage. A *nonzero* g_* consistent across CMB and LSS would favor SC.

2. **Non-Gaussian shape deformations.** Beyond single-field templates, SC admits angle-dependent sound speeds and director mixing, producing shape functions with controlled anisotropic features:

$$f_{\rm NL}^{\rm eff}({\rm sq/iso}) \sim \mathcal{O}\left(\frac{1-c_s^2}{c_s^2}\right) + \mathcal{O}(g_*). \tag{12.16}$$

A squeezed-limit consistent with Eq. (12.8) but with small directional modulations is a distinctive signature.

Table 12.1: Diagnostic comparison between ΛCDM and Spectral Cosmology (SC). "Outcome" indicates which framework is favored given a hypothetical measurement, assuming all gates pass.

Observable	**ΛCDM expectation**	**SC prediction (robust core)**	**Outcome**
g_* (CMB/LSS anisotropy)	$g_* = 0$	$\lvert g_*\rvert \sim \mathcal{O}(\lvert c_\parallel - c_\perp\rvert) \ll 1$	$g_* \neq 0 \Rightarrow$ SC↑
Bispectrum shape (squeezed)	Maldacena relation, isotropic	Same relation + small directional modulations	Detection of modulation ⇒ SC↑
r and n_s joint	Model-dependent; often r small	$r = 16\,\epsilon_\eta\, c_s/c_T$, tilt from spectral slow-roll	Either can fit; not decisive alone
Defect signatures	Typically absent	Allowed with quantized charges; small stochastic imprints	Positive robust signal ⇒ SC↑
$w_{\rm eff}(z)$	-1 (constant)	$-1+\delta(z)$, mild time & env. dependence	Significant $\delta(z)$ or env. link ⇒ SC↑
c_T (today)	1	1 (restored)	$c_T \neq 1$ at $z \sim 0$ ⇒ both disfavored
c_T (early)	1	$1+\delta c_T(z \gg 0)$ possible, restored later	Epoch dependence ⇒ SC↑

3. **Defect/soliton sector.** If active, the defect spectrum (strings, walls, Hopf textures) induces *stochastic, scale-selective* contributions to CMB lensing $C_\ell^{\kappa\kappa}$ and small-scale $P(k)$; correlations with halo shapes and alignments reflect the director field $\hat{\chi}$. A positive detection within Gate TQ quantization rules supports SC.

4. **Late-time acceleration correlations.** Residual η-relaxation yields a mildly time-dependent equation of state $w_{\rm eff}(z) = -1 + \delta(z)$ with *environmental* correlations:

$$\delta(z, \mathbf{x}) \approx \delta_0(z) \;+\; \alpha\, \Delta_\rho(\mathbf{x}, z) \;+\; \beta\, \mathcal{I}_\chi(\mathbf{x}, z), \tag{12.17}$$

   where Δ_ρ is a smoothed density field and $\mathcal{I}_\chi$ a director-alignment functional. Cross-correlating low-z acceleration proxies with lensing maps can discriminate SC from a rigid Λ.

5. **Tensor speed & phase lags.** Early-time deviations $c_T = 1 + \delta c_T(\mathcal{S})$ at high redshift with restoration at late times produce *scale- or epoch-dependent* imprints consistent with multi-messenger bounds today. Detection of such a pattern would favor SC.

III. Failure modes (falsifiers)

- **Gate failures:** ghosts/gradient instabilities; $|c_T - 1|$ exceeding multi-messenger bounds at low z; violation of soft theorems or Ward identities; topological charges not quantized.
- **Data failures:** (i) $\{n_s, r, f_{\rm NL}\}$ jointly outside current allowed regions; (ii) absence of any permitted region once all late-time probes (BAO+SNe+RSD+lensing+GW) are combined; (iii) environmental correlations of $w_{\rm eff}$ *ruled out* at high significance.
- **Numerical/structural failures:** instability of integrators within the admissible region; non-reproducible synthetic pipelines; inconsistent restoration to GR-like propagation ($c_T \not\to 1$).

Decision table (diagnostic view)

Decision logic (flow)

Given data vector $\mathbf{D}$,

1. If any hard gate fails ⇒ **SC excluded**.
2. Else, compute discriminants $\{g_*,\ f_{\rm NL}^{\rm eff}(\text{shape}),\ \text{defect metrics},\ \delta(z),\ c_T(z)\}$.

3. If *all* discriminants are null within errors and a ΛCDM point gives a strictly better global likelihood with comparable complexity, **ΛCDM preferred**.

4. If one or more discriminants show statistically significant non-null values consistent with SC maps and inconsistent with ΛCDM, **SC favored**.

Summary. SC is designed to be *indistinguishable* from ΛCDM in its restoration regime, while offering *specific, testable* departures tied to its spectral microphysics. Table 12.1 and the flow above provide an operational rubric for deciding where SC wins or fails.

12.3 Definitive, Falsifiable Signatures

Purpose. We list a *short, prioritized* set of signatures that (i) are robust consequences of the spectral framework, (ii) have *null* predictions in ΛCDM (or sharply different scaling), and (iii) admit concrete detection thresholds with current/near–term data. Each item specifies: observable, SC prediction (with formula), null test, target sensitivity, and primary datasets.

S1. Primordial statistical anisotropy (quadrupolar modulation)

$$\mathcal{P}_\zeta(\mathbf{k}) = \mathcal{P}_\zeta^{(0)}(k)\Big[1 + g_*\,(\hat{\mathbf{k}}\cdot\hat{\chi}^{(0)})^2\Big], \qquad |g_*| \sim \mathcal{O}\big(|c_\parallel - c_\perp|\big) \ll 1. \tag{12.18}$$

Null in ΛCDM: $g_* = 0$ (up to systematics). *SC falsifier:* consistent joint constraint $|g_*| < g_*^{\min}$ with $g_*^{\min}$ below the minimal value predicted by any admissible $(c_\parallel - c_\perp)$ within the gates. *Near–term target:* $\sigma(g_*) \lesssim 10^{-3}$ (CMB $\{TT, TE, EE\}$ + LSS angle–resolved $P(k,\mu)$). *Channels:* Planck/ACT/SPT → CMB–S4; DESI/Euclid LSS.

S2. Direction–dependent squeezed–limit non–Gaussianity

Angle–dependent sound speed and director mixing yield a small, anisotropic deformation of the squeezed bispectrum:

$$B_\zeta(\mathbf{k}_L, \mathbf{k}_S, -\mathbf{k}_L - \mathbf{k}_S) = \Big[1 + \beta_{\rm aniso}\,(\hat{\mathbf{k}}_L\cdot\hat{\chi}^{(0)})^2 + \cdots\Big]\, B_\zeta^{\text{single-clock}}, \qquad \beta_{\rm aniso} \sim \mathcal{O}(g_*). \tag{12.19}$$

Null in ΛCDM: isotropic single–field consistency form. *SC falsifier:* isotropic squeezed shape with $|\beta_{\rm aniso}| < \beta_{\min}$ below SC floor across data combinations. *Target:* $\sigma(\beta_{\rm aniso}) \lesssim 10^{-2}$ via CMB+LSS bispectrum tomography. *Channels:* Planck/Simons Obs. + DESI/Euclid bispectra.

S3. Early–epoch tensor speed deviation with late restoration

$$c_T(z) = 1 + \delta c_T(z), \qquad \delta c_T(z \ll 1) \to 0, \quad \delta c_T(z_\star) \neq 0 \text{ allowed (early)}. \tag{12.20}$$

Null in ΛCDM: $c_T \equiv 1$ at all epochs. *SC prediction:* possible $\delta c_T \neq 0$ during the η–wave stage, constrained by restoration gate at late times. *Tests:* (i) CMB tensor transfer–function distortions; (ii) PTA/LISA phase–lag tomography vs. z. *Target:* sensitivity to epoch–dependent δc_T at the 10^{-2} level integrated over the window. *Falsifier:* stringent bound $\delta c_T(z) \simeq 0$ at *all* epochs incompatible with SC curvature sector within gates.

S4. Defect/soliton stochastic imprints (quantized, scale–selective)

Topological sectors (strings/walls/Hopfions) contribute to lensing and small–scale statistics:

$$C_\ell^{\kappa\kappa} = C_\ell^{\kappa\kappa}\big|_{\Lambda\rm CDM} + A_{\rm def}\, S_\ell^{(\rm def)}(Q_{\rm top}), \qquad Q_{\rm top} \in \mathbb{Z} \ \text{(quantized)}. \tag{12.21}$$

Null in ΛCDM: $A_{\rm def} = 0$. *SC prediction:* $A_{\rm def} > 0$ with template $S_\ell^{(\rm def)}$ fixed by §5; correlated halo–shape alignments with the director. *Target:* $\sigma(A_{\rm def})/A_{\rm def} \lesssim 0.5$ (first detection), plus integer–valued $Q_{\rm top}$ consistency. *Falsifier:* Non–detection with upper bound below SC floor derived from admissible freeze–out scales ξ.

S5. Environmental modulation of late–time acceleration

Residual η–relaxation induces a weak correlation between the effective equation–of–state perturbation and environment:

$$\delta w_{\rm eff}(\mathbf{x}, z) \equiv w_{\rm eff}(\mathbf{x}, z) + 1 = \alpha(z)\,\Delta_\rho(\mathbf{x}, z) + \beta(z)\,\mathcal{I}_\chi(\mathbf{x}, z) + \cdots . \tag{12.22}$$

Null in ΛCDM: $\delta w_{\rm eff} \equiv 0$ (no environment dependence). *Statistic:* cross–correlation $C_\ell^{\delta w\times\kappa}$ and $C_\ell^{\delta w\times\delta_g}$. *Target:* detect $|\alpha| \gtrsim 10^{-2}$ on degree scales; or constrain below SC floor. *Falsifier:* stringent non–detection across multiple tracers excluding the admissible (α, β) region.

S6. LSS bispectrum with director–induced angular kernel

Beyond gravity–only kernels, SC predicts a small director–dependent correction:

$$B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = B_{\rm grav} + \gamma_{\rm dir} \sum_{i<j} \left[(\hat{\mathbf{k}}_i \cdot \hat{\chi}^{(0)})(\hat{\mathbf{k}}_j \cdot \hat{\chi}^{(0)}) \right] F(k_i, k_j), \tag{12.23}$$

with $\gamma_{\rm dir} \propto g_*$. *Null in ΛCDM:* $\gamma_{\rm dir} = 0$. *Target:* $\sigma(\gamma_{\rm dir}) \lesssim 10^{-3}$ in Euclid/DESI bispectrum multipoles. *Falsifier:* global upper bound below SC–implied floor from (12.18).

S7. Consistency of (n_s, r) with spectral slow–roll map

SC links (n_s, r) to spectral parameters $(\epsilon_\eta, \eta_\eta, c_s/c_T)$:

$$n_s - 1 \simeq -2\epsilon_\eta - \eta_\eta - s_c, \qquad r \simeq 16\,\epsilon_\eta\,\frac{c_s}{c_T}. \tag{12.24}$$

Discriminator: a measured pair (n_s, r) that *cannot* be achieved by any admissible $(\epsilon_\eta, \eta_\eta, c_s/c_T)$ passing the gates falsifies SC (or forces a curvature–sector revision). *Target:* $\sigma(r) \lesssim 10^{-3}$ (Stage–IV CMB); $\sigma(n_s) \sim 10^{-3}$.

S8. Soft theorems with anisotropic corrections

Soft limits for large–scale modulations obey:

$$\lim_{k_L\to 0} \frac{B_\zeta(\mathbf{k}_L, \mathbf{k}_S, -\mathbf{k}_L - \mathbf{k}_S)}{P_\zeta(k_L)P_\zeta(k_S)} = -(n_s - 1) + \mathcal{O}(g_*\,\cos^2\theta) + \cdots . \tag{12.25}$$

Null in ΛCDM: purely isotropic RHS. *Test: angle–conditioned* squeezed–limit estimator; detect $\cos^2\theta$ term or bound it below SC floor.

A compact decision table

Summary. Items S1–S8 define a compact, falsifiable interface between spectral cosmology and data. Each has a precise *null* in ΛCDM and a calculable SC *floor* fixed by gates and by the inflationary η–wave microphysics. A single decisive non–detection below the floor, or an inconsistent (n_s, r) pair outside the SC manifold, would falsify the framework; conversely, a coherent pattern across these signatures would be difficult to mimic with ΛCDM extensions and would point to a spectral origin of cosmic structure.

Table 12.2: Prioritized falsifiable signatures. "Floor" is the minimal SC signal implied by admissible parameters within gates; detection below the floor falsifies SC or a sector thereof.

Signature	**SC prediction (floor)**	**Null in ΛCDM**	**Primary datasets**
g_* (primordial anisotropy)	$\lvert g_*\rvert \gtrsim 10^{-3}$ if $\lvert c_\parallel - c_\perp\rvert \gtrsim 10^{-3}$	$g_* = 0$	CMB–S4, DESI/Euclid
Anisotropic squeezed $f_{\rm NL}$	$\lvert\beta_{\rm aniso}\rvert \gtrsim 10^{-2} g_*$	isotropic	CMB+LSS bispectrum
Early $\delta c_T(z)$	epoch–dependent $\lvert\delta c_T\rvert \lesssim 10^{-2}$, restored later	none	CMB tensors, PTA/LISA
Defect imprints ($A_{\rm def}$)	nonzero with quantized $Q_{\rm top}$	none	CMB lensing, LSS small–scale
Env. $\delta w_{\rm eff}$	$\lvert\alpha\rvert \gtrsim 10^{-2}$ on degree scales	zero	SN/BAO+WL cross
LSS angular kernel $\gamma_{\rm dir}$	$\sim g_*$–scaled anisotropic piece	zero	DESI/Euclid bispectrum
(n_s, r) map	must lie on SC manifold (12.24)	generic	CMB–S4/BICEP
Soft theorems (anisotropic)	$\cos^2\theta$ correction $\propto g_*$	none	Multi–survey squeezed estimators

Part VI

Causality, Time, and Topology in Spectral Cosmology

Chapter 13

Arrow of Time, Emergent Fabric, and Nontrivial Topologies

13.1 Arrow of Time: Emergence, Fragility, and Fluctuation Theorems

Operational split: temporal ordering vs. temporal direction

In the spectral framework the *existence* of time and its *arrow* have distinct origins:

1. **Temporal ordering (clock):** supplied by the internal scalar T once dynamical channels activate. It equips histories with an ordering parameter along the aether flow u^μ via $d\tau \propto u_\mu\, dX^\mu$.

2. **Temporal direction (arrow):** selected by the spectral viscosity $\eta > 0$ through irreversible production of spectral entropy. With $\eta = 0$ the dynamics can be (locally) time-reversal invariant even in the presence of a well-defined clock.

Throughout, the effective metric $g^{\rm eff}_{\mu\nu}(S)$ and the aether frame $u^\mu(S)$ are the emergent background fields constructed from the structural state $S = \{\Phi, \rho, \eta, \chi^\mu, T\}$; covariant derivatives are $\nabla_\mu = D_\mu + \omega_\mu$ as in Vol. I.

Spectral entropy current and the second law

Define the spectral entropy current $J^\mu_s \equiv s\, u^\mu$ with production rate

$$\nabla_\mu J^\mu_s \;\equiv\; \sigma_{\rm diss} \;=\; 2\,\eta\, g^{\mu\nu}_{\rm eff}\,(\nabla_\mu\Phi)^*(\nabla_\nu\Phi) \;+\; \eta\,\mathcal{Q}[\rho,\chi;\nabla S] \;\geq 0, \tag{13.1}$$

where $\mathcal{Q}$ is a positive semi-definite quadratic form built from structural gradients (explicit in Vol. I, Ch. 3). Integrating over a spacelike region Σ_t with unit normal $n_\mu \parallel u_\mu$,

$$\frac{d}{dt}\int_{\Sigma_t}\sqrt{\gamma}\, s \;=\; \int_{\Sigma_t}\sqrt{\gamma}\,\sigma_{\rm diss} \;-\; \oint_{\partial\Sigma_t} d\Sigma_\mu\, J^\mu_s, \tag{13.2}$$

so that local entropy can decrease only if compensated by outward entropy flux through $\partial\Sigma_t$.

A Lyapunov functional for the arrow

Let $\mathcal{H}_{\rm coh}(S)$ denote the coherent (non-dissipative) energy density and $\theta(\eta)$ a monotone "spectral temperature" functional with $\theta'(\eta) > 0$. Define

$$\mathcal{L}_{\rm spec}[S;\Sigma_t] \;\equiv\; \int_{\Sigma_t}\sqrt{\gamma}\,\Big[\mathcal{H}_{\rm coh}(S) - \theta(\eta)\, s\Big]. \tag{13.3}$$

Proposition 1 (Monotonicity). Assuming (i) the Rayleigh/GKSL structure of dissipation for the reduced aether patch and (ii) the no-ghost/hyperbolicity gates of Vol. I, one has

$$\frac{d}{dt}\,\mathcal{L}_{\rm spec}[S;\Sigma_t] \;\le\; -\int_{\Sigma_t}\sqrt{\gamma}\,\Big[\Gamma_\eta\,\dot\eta^{\,2} \;+\; 2\,\eta\,g^{\mu\nu}_{\rm eff}(\nabla_\mu\Phi)^*(\nabla_\nu\Phi) \;+\; \eta\,\mathcal{Q}\Big] \;+\; \underbrace{\oint_{\partial\Sigma_t} d\Sigma_\mu\,\theta(\eta)\,J^\mu_s}_{\text{boundary influx}}, \tag{13.4}$$

with $\Gamma_\eta(\mathcal{S}) \ge 0$ the Rayleigh coefficient. Hence $\mathcal{L}_{\rm spec}$ is a Lyapunov functional modulo boundary influx; in closed systems it is nonincreasing.

Sketch of proof. Vary (13.3) along solutions of the Euler–Lagrange equations with Rayleigh term $\frac{1}{2}\Gamma_\eta(u^\mu\nabla_\mu\eta)^2$. The coherent part is conserved up to the work done by dissipators; the $-\theta s$ term decreases by (13.1) after choosing $\theta'(\eta)\ge 0$. Boundary terms deliver the flux contribution. □

Fragility windows and local arrow weakening

Equation (13.2) admits transient local entropy decrease if

$$\int_{\Sigma_t}\sqrt{\gamma}\,\sigma_{\rm diss} \;<\; \oint_{\partial\Sigma_t} d\Sigma_\mu\,J^\mu_s, \qquad \text{with} \quad \eta \ll \eta_{\rm bg},\;\; |\nabla S| \text{ coherent and small.} \tag{13.5}$$

We term such conditions *fragility windows*. They require (i) reduced dissipation η in the patch, (ii) strong entropy outflow to the environment, and (iii) maintenance of hyperbolicity (no superluminal cones). These episodes underlie recoherence phenomena without violating the global second law.

Spectral fluctuation theorems

Consider a coarse-grained Markovian evolution for a set of slow aether observables X governed by a GKSL generator $\mathcal{L}$ that satisfies local detailed balance at spectral temperature $\theta(\eta)$:

$$\partial_t P(X,t) = \mathcal{L}^\dagger P, \qquad \frac{W_{\rm fwd}(X\to X')}{W_{\rm rev}(\Theta X'\to\Theta X)} = \exp\Big[\Delta\Sigma(X\to X')\Big], \tag{13.6}$$

with Θ the kinematic time-reversal on X and $\Delta\Sigma$ the path-wise entropy production. Then the Crooks/Jarzynski-type relations hold:

$$\frac{\mathcal{P}_{\rm fwd}(\Delta\Sigma)}{\mathcal{P}_{\rm rev}(-\Delta\Sigma)} = e^{\Delta\Sigma}, \qquad \big\langle e^{-\Delta\Sigma}\big\rangle_{\rm fwd} = 1, \qquad \langle\Delta\Sigma\rangle_{\rm fwd} \ge 0, \tag{13.7}$$

tying arrow selection to the statistics of rare negative-$\Delta\Sigma$ excursions. In the spectral medium these relations quantify the probability of arrow-weakening events inside fragility windows, while ensuring global irreversibility.

Gate formulation for the arrow

We summarize the arrow-related gates used in Vol. II:

- **(A1) Ordering gate:** T defines a good clock in the aether frame: $u^\mu\nabla_\mu T > 0$.
- **(A2) Positivity gate:** $\sigma_{\rm diss} \ge 0$ and $\Gamma_\eta \ge 0$ everywhere in the inflationary and late-time regimes.
- **(A3) Causal gate:** principal cones of all sectors lie within those of $g_{\rm eff}$; $0 < c_s^2(\theta), c_T^2 \le 1$.
- **(A4) Fragility control:** arrow-weakening windows satisfy (13.5) without violating (A2–A3).
- **(A5) Restoration:** post-inflation, $c_T \to 1$ and anisotropies decay to observational bounds.

These gates render the emergence and stability of the arrow a *testable* output rather than a postulate: violations map to specific parameter regions in $(\eta, \Gamma_\eta, \kappa_{\rm stiff}, \dots)$ to be excluded by data.

Summary. The clock field T *enables* temporal order; the viscosity field η *selects* temporal direction by enforcing a monotone Lyapunov flow up to boundary fluxes. Fragility windows, captured by (13.5), explain localized recoherence without endangering global irreversibility, while fluctuation theorems (13.7) quantify arrow fluctuations in finite spectral patches. This operational decomposition will be crucial when we discuss nontrivial topologies and chronology protection in Sec. 13.3.

13.2 Emergent Fabric: Effective Metric, Causal Cones, and Hyperbolicity

From structural state to an effective causal geometry

Given the structural state $S(x) = \{\Phi, \rho, \eta, \chi^\mu, T\}$, we construct an orthonormal coframe $e^a{}_\mu(S)$ and the effective Lorentzian metric (Vol. I)

$$g^{\text{eff}}_{\mu\nu}(S) \;=\; \eta_{ab}\, e^a{}_\mu e^b{}_\nu, \qquad \eta_{ab} = \text{diag}(-1, 1, 1, 1), \tag{13.8}$$

together with the aether frame $u^\mu(S)$ (unit, timelike with respect to g^{eff}). The covariant derivative acting on any field multiplet is $\nabla_\mu \equiv D_\mu + \omega_\mu$, where D_μ is the spectral derivative and ω_μ the emergent connection (Vol. I, Chs. 2–3).

Characteristic cones from the principal symbol

Linearizing the unified action about a background S_0 yields a quadratic Lagrangian

$$\mathcal{L}^{(2)} = \tfrac{1}{2}\, \delta\mathbf{S}^\top \left[\mathsf{K}\, \partial_t^2 + \mathsf{C}\, \partial_t + \mathsf{Z}^{ij}\, \nabla_i \nabla_j + \mathsf{M}^2\right] \delta\mathbf{S}, \tag{13.9}$$

with positive time–kinetic block $\mathsf{K} \succ 0$ (no ghosts) and spatial principal block Z^{ij}. For a covector $\xi_\mu = (\omega, \mathbf{k})$ the principal symbol is

$$\mathscr{P}(\xi) \;\equiv\; \omega^2\, \mathsf{K} \;+\; \mathsf{Z}^{ij} k_i k_j, \qquad \det \mathscr{P}(\xi) = 0 \quad \Longleftrightarrow \quad \text{characteristics}. \tag{13.10}$$

Each propagating sector α defines a *characteristic cone* C_α in cotangent space by $\det \mathscr{P}_\alpha(\xi) = 0$. We impose the *causal sandwich* gate

$$C_\alpha \;\subseteq\; C\big(g^{\text{eff}}\big) \qquad \text{for all sectors } \alpha, \tag{13.11}$$

meaning no mode outruns the null cone of g^{eff}. In an orthonormal frame adapted to u^μ and the director $\hat{\chi}^{(0)}$, dispersion takes the form

$$\omega_\alpha^2(\mathbf{k}) \;=\; c_{\parallel,\alpha}^2\, k_\parallel^2 \;+\; c_{\perp,\alpha}^2\, k_\perp^2 \;+\; m_{\alpha,\text{eff}}^2 \;+\; \cdots, \qquad 0 < c_{\perp,\alpha}^2 \le c_{\parallel,\alpha}^2 \le 1, \tag{13.12}$$

with anisotropy inherited from $\hat{\chi}^{(0)}$ (Vol. I, Eq. (2.73)–(2.79) analogues).

Strong hyperbolicity and microcausality

Let n_i be any spatial unit covector. *Strong hyperbolicity* holds if there exists a symmetrizer $\mathsf{H} \succ 0$ such that HK and $\mathsf{HZ}^{ij} n_i n_j$ are symmetric and the eigenvalues of $\mathsf{K}^{-1}\mathsf{Z}^{ij} n_i n_j$ are real and diagonalizable. Then the Cauchy problem is well-posed and domains of dependence are bounded by the union of sector cones C_α. Combined with the sandwich condition (13.11) this yields *microcausality* with respect to g^{eff}: no signal, classical or quantum (in the EFT sense), exits the effective light cone.

Stable causality and chronology

A spacetime $(\mathcal{M}, g^{\text{eff}})$ is *stably causal* if there exists a smooth time function τ whose gradient is everywhere timelike:

$$g^{\text{eff}}_{\mu\nu}\, \nabla^\mu \tau\, \nabla^\nu \tau \;<\; 0. \tag{13.13}$$

In the spectral setting we take $\tau = f(T)$ with $f' > 0$. The *ordering gate* (A1) of Sec. 13.1 demands $u^\mu \nabla_\mu T > 0$. If, in addition, the level sets Σ_τ are Cauchy hypersurfaces for the hyperbolic system (13.9), then $(\mathcal{M}, g^{\text{eff}})$ is globally hyperbolic and admits no closed timelike curves (CTCs). Chronology violation would require either (i) breakdown of stable causality (e.g. loss of timelike $\nabla_\mu T$) or (ii) sector cones escaping the gravitational cone (violation of (13.11)). Both are excluded by the gates below.

Spectral energy conditions and focusing

Define the effective stress tensor $T^{\text{eff}}_{\mu\nu}$ (Vol. I, Ch. 3). For any null ℓ^μ of g^{eff},

$$T^{\text{eff}}_{\mu\nu}\,\ell^\mu\ell^\nu \;=\; \sum_\alpha \left[\mathcal{Z}_\alpha(\nabla_\parallel\varphi_\alpha)^2 + \tilde{\mathcal{Z}}_\alpha(\nabla_\perp\varphi_\alpha)^2\right] \;+\; (\text{potential terms}) \;\geq 0, \tag{13.14}$$

provided the time–kinetic matrices are positive and the gradient blocks $\mathcal{Z}_\alpha, \tilde{\mathcal{Z}}_\alpha$ are positive semi-definite. This *spectral null energy condition* (sNEC) implies the Raychaudhuri focusing inequality for the aether congruence

$$\dot\theta \;=\; -\tfrac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - \frac{1}{2M^2_{\text{eff}}}\,(\rho_{\text{tot}} + p_{\text{tot}}) \;\leq\; 0 \,+\, \omega^2, \tag{13.15}$$

with $\theta = \nabla_\mu u^\mu$. During inflation and at late times ω^2 is constrained to be small by isotropy bounds, ensuring controlled focusing/defocusing compatible with observed expansion.

Observational and theoretical superluminality tests

The causal sandwich (13.11) and the speed bounds (13.12) map to concrete constraints:

- **GW speed:** $c_T^2 = 1 + \delta c_T(\mathcal{S})$ with $|\delta c_T| \ll 1$ (multi-messenger bounds).
- **Vacuum Čerenkov:** any sector with $c_\alpha < 1$ forbids charged primaries emitting that sector; conversely $c_\alpha > 1$ would induce anomalous losses. UHECR bounds constrain $c_\alpha - 1$.
- **Photon sector proxy:** in the emergent gauge sector (Vol. I, Ch. 8) the effective photon cone must coincide with (or be inside) $C(g^{\text{eff}})$ to avoid arrival-time anomalies.
- **Birefringence/anisotropy:** differences $c_{\parallel,\alpha} - c_{\perp,\alpha}$ are limited by CMB/LSS isotropy; they tie back to the director stiffness parameters in Z^{ij}.

Causal gate set for the emergent fabric

We collect the working gates for this section:

$$\textbf{(C1) Sandwich:}\quad C_\alpha \subseteq C(g^{\text{eff}}) \;\;\forall\alpha. \tag{13.16}$$

$$\textbf{(C2) Strong hyperbolicity:}\quad \exists\,\mathsf{H} \succ 0 \text{ symmetrizer for } \mathsf{K}, \mathsf{Z}^{ij}. \tag{13.17}$$

$$\textbf{(C3) Stable causality:}\quad \tau = f(T),\;\; f' > 0,\;\; g^{\text{eff}}(\nabla\tau, \nabla\tau) < 0 \text{ globally.} \tag{13.18}$$

$$\textbf{(C4) sNEC:}\quad T^{\text{eff}}_{\mu\nu}\ell^\mu\ell^\nu \geq 0 \;\;\forall\; \ell^2 = 0 \text{ of } g^{\text{eff}}. \tag{13.19}$$

$$\textbf{(C5) Observational:}\quad |c_T - 1|,\;\; |c_{\parallel,\alpha} - c_{\perp,\alpha}| \text{ within current bounds.} \tag{13.20}$$

Passing (13.16)–(13.20) guarantees a coherent, testable causal structure with no superluminal leaks and no chronology violation within the regime of validity of the effective description.

Summary. The emergent fabric is the *causal carrier* of the spectral medium: its null cone bounds all sector cones, strong hyperbolicity enforces deterministic evolution, and stable causality excludes CTCs. Anisotropies and speed deviations are small, computable functions of structural couplings, tightly constrained by observations. This foundation enables a disciplined discussion of nontrivial topologies—wormholes, handles, and potential chronology issues—in Sec. 13.3.

13.3 Wormholes, Topology Change, and Chronology: Spectral Energy Conditions and Censorship

Traversable throats and the flaring–out condition

Let $(\mathcal{M}, g^{\text{eff}})$ be the emergent spacetime obtained from the structural state $S(x) = \{\Phi, \rho, \eta, \chi^\mu, T\}$. A (putative) static, spherically symmetric traversable throat may be charted by

$$ds^2_{\text{eff}} \;=\; -e^{2\Phi_w(r)}\,dt^2 \;+\; \frac{dr^2}{1 - b(r)/r} \;+\; r^2 d\Omega^2, \tag{13.21}$$

with shape function $b(r)$ satisfying $b(r_0) = r_0$ at the throat $r = r_0$ and the flaring–out condition $\left[b'(r_0) - b(r_0)/r_0\right] < 0$. Projecting the effective Einstein equation of Vol. I, Ch. 3 along radial null vectors k^μ at the throat yields

$$T^{\text{eff}}_{\mu\nu} k^\mu k^\nu \Big|_{r_0} \;=\; \frac{1}{8\pi M^2_{\text{eff}}} \frac{b'(r_0) - 1}{r_0^2} \;<\; 0, \tag{13.22}$$

so any *traversable* throat demands a local violation of the (null) energy condition (NEC) with respect to g^{eff}.

Spectral NEC/ANEC from positivity and dissipation

In the spectral medium, the effective stress tensor can be decomposed into sectoral contributions (Vol. I) $T^{\text{eff}}_{\mu\nu} = \sum_\alpha T^{(\alpha)}_{\mu\nu}$ with kinetic matrices positive (no–ghost gate) and Rayleigh/GKSL dissipation nonnegative. For any null vector ℓ^μ of g^{eff} we defined in Eq. (13.14) the *spectral NEC* (sNEC) $T^{\text{eff}}_{\mu\nu} \ell^\mu \ell^\nu \geq 0$. Beyond pointwise positivity, dissipation and Kubo–type positivity imply an *averaged* inequality along complete null geodesics γ:

$$\int_{-\infty}^{+\infty} d\lambda \; T^{\text{eff}}_{\mu\nu} \, k^\mu k^\nu \;\geq\; 0, \qquad k^\mu = \frac{dx^\mu}{d\lambda} \text{ null geodesic of } g^{\text{eff}}, \tag{13.23}$$

the spectral ANEC, up to exponentially small transients near phase boundaries. The integral receives strictly nonnegative contributions from dissipative channels proportional to $\eta > 0$.

Spectral Topological Censorship (STC): statement and proof sketch

We work in an asymptotically simple region $\mathcal{O} \subset \mathcal{M}$ (asymptotically FRW/de Sitter or asymptotically flat in suitable patches) that admits a conformal boundary representing the observational domain of a comoving observer family.

Theorem (Spectral Topological Censorship (STC))**.** *Assume: (i) $(\mathcal{O}, g^{\text{eff}})$ is globally hyperbolic and stably causal with time function $\tau = f(T)$ (Sec. 13.2); (ii) the causal sandwich holds, $C_\alpha \subseteq C(g^{\text{eff}})$ for all sectors; (iii) the spectral ANEC (13.23) is satisfied along all complete null geodesics in $\mathcal{O}$; (iv) the boundary of $\mathcal{O}$ is weakly asymptotically simple (no caustic pathologies at infinity). Then any causal curve from $\partial^-\mathcal{O}$ to $\partial^+\mathcal{O}$ is homotopic to a curve in a simply–connected neighborhood; in particular, traversable wormhole handles cannot be probed by causal signals launched and received in $\mathcal{O}$.*

Proof sketch. Consider an achronal null geodesic congruence connecting the boundary components. The Raychaudhuri equation for the expansion θ of the congruence (Eq. (13.15)) together with shear positivity and the sANEC (13.23) imply the absence of expansion–free, noncontractible "shortcuts": negative focusing required to thread a topology–nontrivial channel would necessitate $\int T^{\text{eff}}_{\mu\nu} k^\mu k^\nu d\lambda < 0$, contradicting (13.23). Global hyperbolicity (Cauchy development of Σ_τ) then yields the homotopy trivialization of causal curves. □

Consequences. Within the validity of the effective description, *traversable* wormholes are ruled out unless one of the gates is broken: loss of stable causality, violation of spectral ANEC, or sector cones outrunning the gravitational light cone.

Chronology protection in the spectral medium

Closed timelike curves (CTCs) require either (i) breakdown of stable causality (no global τ with timelike gradient), or (ii) superluminal leakage of some sector that allows effective timelike shortcuts when projected onto g^{eff}. The gate set of Sec. 13.2 prevents (ii) by enforcing the causal sandwich and observational speed bounds ($c_T \simeq 1$, $c_{\parallel,\alpha} - c_{\perp,\alpha} \ll 1$). Condition (i) would require $u^\mu \nabla_\mu T \leq 0$ somewhere (ordering collapse) or a nonintegrable twist strong enough to invalidate $\tau = f(T)$. Both are incompatible with the entropy gate ($\sigma_{\text{diss}} \geq 0$) in sustained macroscopic phases. We therefore obtain a *spectral chronology–protection principle*: CTCs cannot be dynamically generated in any phase obeying the gates.

On topology change and handle nucleation

Geroch–type no–go arguments in GR forbid topology change in globally hyperbolic spacetimes with "reasonable" energy conditions. In the spectral setting:

- **Pre–ignition epoch.** Before the emergence of time ordering (quiescent void), notions of causal topology do not apply; nontrivial spatial topology may be *selected* at ignition without contradiction, since no causal structure is yet defined.
- **Post–ignition phases.** After $\tau = f(T)$ exists and the gates hold, topology change would require singularities or violations of sANEC. "Handle nucleation" could in principle occur via instanton–like events in highly excited regimes, but the associated action S_{inst} is large and the event would entail a localized breakdown of the effective description or of sANEC; thus it is exponentially suppressed and outside the trusted EFT regime.

Possible loopholes and why they close

1. **Localized NEC dips from dispersion.** Higher–derivative spectral operators may allow brief, localized $T^{\text{eff}}_{\mu\nu}k^\mu k^\nu < 0$. However, GKSL positivity and dissipation imply *quantum–inequality–type* bounds, ensuring ANEC (13.23) for achronal geodesics; small, bounded "dips" cannot sustain a traversable throat.
2. **Strong anisotropy funnels.** Large $c_\parallel - c_\perp$ might mimic superluminal channels. CMB/LSS isotropy and GW arrival times severely constrain this; violating them breaks gate (C5).
3. **Exotic boundary conditions.** Wormhole traversability often relies on special asymptotics. In cosmological domains with horizons, STC still applies to the observer's domain of dependence; nontrivial topology may exist "beyond" but remains causally censored.

Gate set for exotic topology and chronology

We summarize operational requirements:

$$\textbf{(T1) sNEC/ANEC:}\quad T^{\text{eff}}_{\mu\nu}\ell^\mu\ell^\nu \geq 0, \quad \int T^{\text{eff}}_{\mu\nu}k^\mu k^\nu d\lambda \geq 0. \tag{13.24}$$

$$\textbf{(T2) Sandwich \& speeds:}\quad C_\alpha \subseteq C(g^{\text{eff}}),\ \ |c_T - 1|,\ |c_{\parallel,\alpha} - c_{\perp,\alpha}| \text{ small.} \tag{13.25}$$

$$\textbf{(T3) Stable causality:}\quad \exists\, \tau = f(T),\ g^{\text{eff}}(\nabla\tau, \nabla\tau) < 0 \text{ globally.} \tag{13.26}$$

$$\textbf{(T4) Global hyperbolicity:}\quad \text{Cauchy slices } \Sigma_\tau \text{ exist; no causal edges in } \mathcal{O}. \tag{13.27}$$

$$\textbf{(T5) Dissipation positivity:}\quad \sigma_{\text{diss}} \geq 0 \text{ (entropy gate), precluding sustained NEC violation.} \tag{13.28}$$

Passing (13.24)–(13.28) *censors* traversable wormholes and forbids chronology violation in the emergent–metric regime.

Summary. In the Quantum–Aether framework, the very mechanisms that generate the effective geometry—positive kinetic energies, bounded cones, and irreversible spectral dissipation—conspire to enforce a *topological censorship*: any exotic topology is either (i) pre–causal (selected before time emerges) or (ii) causally hidden once the gates are active. Traversable wormholes and time machines would signal a breakdown of at least one gate and thus fall outside the controlled regime of the theory.

13.4 Numerical Thought Experiments and Protocols

Objectives and control parameters

We propose a suite of reproducible, end–to–end numerical experiments designed to *stress–test* the causal and topological claims of Secs. 13.1–13.3. Each experiment evolves the coupled structural

fields $S = \{\Phi, \rho, \eta, \chi^\mu, T\}$ together with the emergent geometry $g^{\text{eff}}[S]$ in a finite domain $\Omega \subset \mathbb{R}^3$ with absorbing boundaries. We scan non–dimensional controls

$$\epsilon_\eta := \frac{\kappa_\eta \dot{\bar{\eta}}^2}{2V_\eta}, \quad \alpha_{\text{aniso}} := \frac{|c_\parallel - c_\perp|}{\max(c_\parallel, c_\perp)}, \quad \beta_{\text{diss}} := \frac{\Gamma_\eta}{3H\kappa_\eta}, \quad \zeta_\chi := \|\nabla\times\chi\|/\|\nabla\chi\|,$$

and, when relevant, a wormhole–like *shape forcing* amplitude $\mathcal{F}_b$ (defined below).

Model evolution equations (schematic PDE system)

We integrate a first–order–in–time, mixed hyperbolic–parabolic system

$$\partial_t U = \mathcal{A}^i(U)\,\partial_i U + \mathcal{B}^{ij}(U)\,\partial_i\partial_j U + \mathcal{S}(U), \qquad U \equiv (\Phi, \rho, \eta, \chi^I, T, e^a{}_\mu, \omega_\mu, \ldots), \tag{13.29}$$

$$G_{\mu\nu}[g^{\text{eff}}] = M_{\text{eff}}^{-2}(\rho)\,T^{\text{eff}}_{\mu\nu}[U] \quad \text{(Palatini–type closure)}, \tag{13.30}$$

where $\mathcal{A}^i$ encodes the principal, cone–defining part (characteristic speeds $c_{\parallel,\perp}, c_T$), $\mathcal{B}^{ij}$ aggregates dissipative/regularizing pieces (Rayleigh, GKSL), and $\mathcal{S}$ collects potentials and algebraic constraints. Gauge choices: comoving–aether ($\delta T = 0$) and generalized harmonic reductions for g^{eff} when evolving Eq. (13.30). Time stepping obeys a CFL bound set by the *largest* sectoral speed:

$$\Delta t \le \text{CFL} \min_{\mathbf{x}\in\Omega} \frac{\Delta x}{\max\{c_T,\ c_{\parallel,\alpha},\ c_{\perp,\alpha}\}}.$$

Diagnostics and gate monitors

At each step we record:

(D1) Entropy production: $\sigma_{\text{diss}} = 2\,\eta\, g^{\mu\nu}_{\text{eff}}(\nabla_\mu\Phi)^*(\nabla_\nu\Phi) \ \ge 0,$ (13.31)

(D2) Cone sandwich residual: $\Delta c := \max_\alpha \big(\max\{c_{\parallel,\alpha}, c_{\perp,\alpha}\} - c_T\big)_+, \quad \Delta c = 0$ passes, (13.32)

(D3) Hyperbolicity: $\lambda_{\min}\big(\mathsf{K}\big) > 0,$ and principal symbol eigenvalues real, (13.33)

(D4) sANEC estimator: $\mathcal{I}_{\text{ANEC}}[\gamma] = \int_\gamma T^{\text{eff}}_{\mu\nu} k^\mu k^\nu\, d\lambda \ \ (\ge 0 \text{ required}),$ (13.34)

(D5) Chronology graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with edges timelike w.r.t. g^{eff}; detect cycles (CTCs) via Tarjan/Kosaraju, (13.35)

(D6) Topology trackers: π_1, π_2, π_3 indicators from lattice maps $n(\mathbf{x})$, minimal–area surface trackers. (13.36)

For (13.34), we integrate along numerically traced null geodesics $k^\mu\nabla_\mu k^\nu = 0$ (Sec. 13.1) seeded across Ω.

Protocol A — Arrow fragility and reversibility tests

Goal: Test whether macroscopic arrow reversal can be engineered without violating gates.

1. *ICs:* near–FRW patch with weak anisotropy $\alpha_{\text{aniso}} \lesssim 10^{-3}$, small ϵ_η, and $\sigma_{\text{diss}} > 0$.

2. *Forcing:* impose a localized, time–antisymmetric impulse on η: $\mathcal{S}_\eta \mapsto \mathcal{S}_\eta + f(t)\,h(\mathbf{x})$ with $\int f(t)\,dt = 0$.

3. *Checks:* monitor σ_{diss} sign, Lyapunov functional $\mathcal{F} = \int \mathcal{H}\, d^3x$, and $\mathcal{I}_{\text{ANEC}}$.

4. *Pass/fail:* Any sustained $\sigma_{\text{diss}} < 0$ or $\mathcal{I}_{\text{ANEC}} < 0$ flags a gate violation. Expected outcome: transient partial recoherence, no macroscopic reversal.

Protocol B — Wormhole driving under spectral gates

Goal: Attempt to nucleate a traversable throat while keeping (13.31)–(13.34) nonnegative.

1. *ICs:* two quasi–asymptotic "mouth" regions separated by a neck seeded in the metric sector.
2. *Shape forcing:* add a compact support source $\mathcal{F}_b$ to the geometric closure to mimic $b(r)$–like deformation:
$$\delta T^{\text{eff}}_{\mu\nu} \propto \mathcal{F}_b \, \nabla_\mu \nabla_\nu \psi(\mathbf{x}), \quad \psi \text{ localized on the neck.}$$
3. *Monitors:* evaluate $T^{\text{eff}}_{\mu\nu} k^\mu k^\nu$ at the putative throat and $\mathcal{I}_{\text{ANEC}}$ along threading nulls.
4. *Pass/fail:* If sANEC holds, numerics exhibit focusing and pinching (non–traversable); traversability requires $\mathcal{I}_{\text{ANEC}} < 0$ (forbidden by gates).

Protocol C — Topology–change attempts (handle nucleation)

Goal: Drive a necking transition and test if causal, smooth topology change is possible.

1. *ICs:* single connected domain; impose a time–dependent "isthmus" in g^{eff}.
2. *Trackers:* minimal–area surfaces $\mathcal{A}_{\min}(t)$, expansion θ of null congruences, curvature invariants.
3. *Gate:* global hyperbolicity and sANEC forbid smooth handle creation; numerics should show either singular pinch–off (leaving domain of validity) or causal censorship.

Protocol D — Cone–tilting and superluminality hunts

Goal: Maximize α_{aniso} and test cone violations.

1. *Ramp:* increase ζ_χ and anisotropic stiffness to push $c_\parallel - c_\perp$.
2. *Monitor:* Δc in (13.32); tensor speed c_T vs. multi–messenger bounds.
3. *Pass:* $\Delta c = 0$ and $|c_T - 1| \ll 1$. *Fail:* any sector outruns c_T (gate breach).

Protocol E — Chronology graph search (CTC detector)

Goal: Algorithmic hunt for closed timelike curves.

1. *Graph build:* discretize spacetime into nodes $v = (\mathbf{x}, t)$. Add a directed edge $v \to v'$ if the straight segment has $g^{\text{eff}}(\dot{x}, \dot{x}) < 0$ (timelike) and lies within the domain of dependence.
2. *Cycle test:* run strongly connected component (SCC) detection; any SCC with a nontrivial cycle signals a candidate CTC.
3. *Gate coupling:* SCC presence must coincide with a gate failure (typically loss of stable causality). In passing runs, the graph is DAG–like (acyclic).

Discretization, stability, and reproducibility

We recommend:

- **Spatial discretization:** 4th–order finite differences (or spectral collocation) with summation–by–parts (SBP) operators; shock–capturing filters only on dissipative subblocks.
- **Time stepping:** IMEX–RK(4,3) with adaptive Δt from CFL and stiffness monitors (spectral radius of $\mathcal{B}^{ij}$).
- **Boundaries:** perfectly matched layers (PML) implemented for each sectoral cone; constraint–preserving outflow for g^{eff}.
- **Validation:** convergence at orders $\{2, 4\}$, conserved charges drift $< 10^{-6}$ over 10^3 steps, identical outcomes across seeds within UQ bands.

Reporting and UQ checklist

Each run must ship:

1. Control vector $(\epsilon_\eta, \alpha_{\rm aniso}, \beta_{\rm diss}, \zeta_\chi, \mathcal{F}_b)$ and discretization tuple $(N_x, N_t, \Delta x, \text{IMEX scheme})$.
2. Gate dashboard: time–series of $\sigma_{\rm diss}$, Δc, $\lambda_{\min}(\mathsf{K})$, $\mathcal{I}_{\rm ANEC}$, SCC count.
3. Uncertainty quantification: mesh refinement study, box–size study, seed ensemble for stochastic forcings, posterior on pass/fail with 95% CIs.

Summary. The protocols of Sec. 13.4 translate the conceptual censorship results into falsifiable, numerical stress–tests. Within the gate–respecting domain, we expect: (i) no macroscopic arrow reversal, (ii) no traversable wormhole formation under sANEC, (iii) no smooth topology change in globally hyperbolic phases, (iv) no CTCs. Any contrary outcome identifies a controlled gate breach and hence a clear target for theoretical repair or empirical refutation.

13.5 Roadmap for Follow-up Work

Purpose

This roadmap translates the conceptual and technical results of Part VII into a sequenced, testable research program. Each milestone lists inputs, concrete deliverables, and *consistency gates* (S/G/TQ/D) that must be passed before advancing.

Phase A : Foundational consolidation

- **A1 — η-wave microphysics:** Finalize the effective sector $\{\kappa_\eta(\mathcal{S}), V_\eta(\eta), \Gamma_\eta(\mathcal{S})\}$ under early-universe gates; derive closed forms for (n_s, r) and bounds on statistical anisotropy.
- **A2 — Background & linear codes:** Release v1 background/perturbation integrators (Sec. 11) with reproducibility tests and unit benchmarks.
- **A3 — Causality pack:** Implement geodesic tracers, ANEC estimators, and chronology graph detectors (Secs. 13.1–13.4).
- **A4 — White paper I:** "Spectral Inflation without an Inflaton" (theory + forecasts).

Gates: S (no ghosts), G (Ward/soft identities), D (cone sandwich, $c_T \simeq 1$), TQ (no forbidden topology change in passing regimes).

Phase B : Nonlinear and phenomenology bridge

- **B1 — Lattice resonance & defects:** Spectral lattice for instability growth, Kibble–Zurek scaling, and dark-soliton statistics (Sec. 5).
- **B2 — Synthetic skies:** End-to-end CMB/LSS/lensing mapmaker with anisotropic imprints (Sec. **??**).
- **B3 — Unified-dark fits (pilot):** Joint likelihoods for $\{\mathcal{P}_\zeta,\ P(k),\ \phi\phi,\ H_0\}$ under late-time η-relaxation (Sec. 9).
- **B4 — White paper II:** "Chladni Cosmology: Web Morphology as a Resonance Pattern."

Gates: S/D (hyperbolicity and subluminality at nonlinear level), G (inclusive IR finiteness), TQ (topological counters consistent with formation without pathology).

Phase C : Confrontation and decision points

- **C1 — Full inference pipelines:** Public Bayesian pipelines (Sec. 10) mapping spectral parameters to $\{n_s, r, f_{\rm NL}, g_*, c_T, w_{\rm eff}(z)\}$.
- **C2 — Multi-messenger constraints:** GW propagation bounds on curvature sector and anisotropy; PTA/LIGO/LISA forecast overlays.
- **C3 — Kill-switch tests:** Decision table (Sec. 12.2): fail fast if any hard contradiction persists (e.g., sustained $c_T \neq 1$, excessive anisotropy, sANEC violation).
- **C4 — White paper III:** "Unified Dark Sector from ρ-structures and η-relaxation."

Gates: All S/G/TQ/D at survey-level precision; restoration checks to GR-like propagation post-inflation.

Open Problems (targeted work packages)

1. **OP1 — Emergent gauge completion:** Rigorous derivation of $SU(3) \times SU(2) \times U(1)$ from fiber isometries with vacuum alignment and t'Hooft anomaly matching.
2. **OP2 — Spectral RG:** Nonperturbative flow for $(M_{\rm eff}, \kappa_\eta, \Gamma_\eta)$ across epochs; universality classes and epochal transitions.
3. **OP3 — WZW/CS & anomaly inflow:** Quantify induced topological terms in the emergent gauge sector subject to cosmological symmetry breaking.
4. **OP4 — sANEC micro-foundations:** Prove sANEC from microscopic dissipation/positivity in the spectral density; classify allowed effective violations (if any).
5. **OP5 — Dark-soliton astrophysics:** Core profiles, self-interactions, lensing and rotation-curve systematics; substructure and tidal signatures.

Deliverables table (living document)

Milestone	Inputs	Outputs	Gates
A2 Codes v1	Vol. I/II EOM	Background+linear solvers	S,D
B1 Lattice	Instability map	Defect stats, ξ, $n_{\rm def}$	S,TQ
B3 Unified fits	Pipelines+data	Posterior on $(w_{\rm eff}, \sigma/M)$	G,D
C2 GW bounds	Curvature sector	Limits on $c_T - 1$	D

Community artifacts and reproducibility

Public repositories (MIT license), versioned datasets, notebooks for all plots/tables, CI tests (unit + UQ), and archived containers ensuring bit-level reproducibility.

Decision logic

Progress to the next phase only if all applicable gates are passed within uncertainties. Hard contradictions trigger redesign of the specific sector rather than ad hoc patches, preserving the pre-geometric ethos.

Part VII

Global Confrontation with Data and Forecasts

Chapter 14

Global Fits, Tension Diagnostics, and Model Selection

14.1 Combined Likelihoods: CMB+LSS+Lensing+GW

Scope. We construct a single hierarchical likelihood that ingests cosmic microwave background (CMB) temperature/polarization, large–scale structure (LSS) clustering and redshift–space distortions (RSD), weak lensing (WL) shear and CMB lensing, and multi–messenger gravitational–wave (GW) propagation constraints. The forward model maps the *spectral–cosmology parameter set*

$$\Theta_{\rm spec} = \big\{V_\eta(\cdot);\ \kappa_\eta,\ \Gamma_\eta;\ M_{\rm eff}(\rho);\ c_T,\ \delta c_T(\mathcal{S});\ c_s(\theta);\ \mathcal{A}_{\rm aniso},\ \hat{\chi}^{(0)};\ \mathcal{D}_{\rm def}\ \text{(defect/soliton sector)};\ w_{\rm eff}(t);\ \cdots\big\}$$

to observables via linear and mildly nonlinear transfer functions derived in Vol. II, Parts I–III, with consistency gates (S/G/TQ/D) acting as hard/soft priors.

Data vector and forward model

Let the joint data vector be

$$\mathbf{d} = \big(\widehat{C}_\ell^{TT},\ \widehat{C}_\ell^{TE},\ \widehat{C}_\ell^{EE},\ \widehat{C}_\ell^{\kappa\kappa},\ \widehat{C}_\ell^{\kappa g_i},\ \widehat{C}_\ell^{g_i g_j};\ \widehat{P}(k_\alpha, z_\beta),\ \widehat{f\sigma_8}(z_\beta);\ \widehat{S}_h(f_a),\ \{\widehat{\Delta t}_n, \widehat{d}_{L,n}\}_{n=1}^{N_{\rm GW}}\big)^T, \tag{14.1}$$

where g_i are tomographic galaxy bins, κ denotes WL or CMB lensing convergence, P and $f\sigma_8$ are LSS 3D summaries, S_h is the (PTA) stochastic background, and $\{\Delta t_n, d_{L,n}\}$ are event–level GW time–of–flight and luminosity–distance inferences.

The theory vector $\mathbf{m}(\Theta, \psi)$ depends on $\Theta \equiv \Theta_{\rm spec}$ and nuisance/calibration parameters ψ (foregrounds, beams, photo–z, shear biases, galaxy bias, selection functions). Schematically,

$$\mathbf{m}(\Theta,\psi) = \Big(C_\ell^{XY}[\Theta] \star \mathsf{M}_\ell^{XY}(\psi)\ +\ T_{\rm fg}^{XY}(\psi),\ \mathsf{W}\circ P(k,z;\Theta,\psi),\ S_h(f;\Theta),\ \{\Delta t(\Theta;z_n),\, p(d_L|\Theta,\psi)\}\Big), \tag{14.2}$$

where $\star$ denotes pseudo-C_ℓ mixing by mask kernels M_ℓ^{XY}, $T_{\rm fg}^{XY}$ are foreground templates, and W applies survey windows and Alcock–Paczynski distortions for LSS.

Joint likelihood and covariance structure

Assuming a quasi–Gaussian summary unless stated otherwise,

$$-2\ln \mathcal{L}_{\rm joint}(\mathbf{d}|\Theta,\psi) = (\mathbf{d}-\mathbf{m})^T \mathbf{C}^{-1}(\mathbf{d}-\mathbf{m}) + \ln|\mathbf{C}| + \text{const}, \tag{14.3}$$

with a block covariance

$$\mathbf{C} = \begin{pmatrix} \mathbf{C}_{\rm CMB} & \mathbf{C}_{\rm X} & \cdots \\ \mathbf{C}_{\rm X}^T & \mathbf{C}_{\rm LSS} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \mathbf{C}_{\rm G} + \mathbf{C}_{\rm NG} + \mathbf{C}_{\rm SSC} + \mathbf{N}. \tag{14.4}$$

Here $\mathbf{C}_{\rm G}$ is the Gaussian (disconnected) part, $\mathbf{C}_{\rm NG}$ includes connected trispectrum terms (important for small–scale LSS/WL), $\mathbf{C}_{\rm SSC}$ captures super–sample covariance (response to background modes δ_b), and $\mathbf{N}$ is noise (including beam/shape/photo-z terms). For angular spectra,

$$\mathrm{Cov}_{\rm G}\Big(\widehat{C}_\ell^{XY}, \widehat{C}_{\ell'}^{WZ}\Big) \simeq \frac{\delta_{\ell\ell'}}{(2\ell+1) f_{\rm sky}} \Big(C_\ell^{XW} C_\ell^{YZ} + C_\ell^{XZ} C_\ell^{YW}\Big)_{\rm eff}, \tag{14.5}$$

$$\mathrm{Cov}_{\rm SSC} \approx \frac{\partial \mathbf{C}}{\partial \delta_b}\, \sigma_b^2 \left(\frac{\partial \mathbf{C}}{\partial \delta_b}\right)^T, \qquad \sigma_b^2 = \int \frac{d^3k}{(2\pi)^3}\, |W_b(\mathbf{k})|^2 P(k), \tag{14.6}$$

with W_b the survey window. For 3D clustering, the Gaussian covariance of $P(k,z)$ in a bin (k_α, z_β) is

$$\mathrm{Cov}\big(\widehat{P}_{\alpha\beta}, \widehat{P}_{\alpha'\beta'}\big) \simeq \frac{2}{N_{\rm m}(\alpha,\beta)} \Big(P_{\alpha\beta} + P_{\rm shot}\Big)^2 \delta_{\alpha\alpha'} \delta_{\beta\beta'}, \tag{14.7}$$

with $N_{\rm m}$ the number of modes and $P_{\rm shot}$ the shot noise.

GW blocks. For $N_{\rm GW}$ sirens,

$$\mathcal{L}_{\rm GW} = \prod_{n=1}^{N_{\rm GW}} p\Big(\widehat{\Delta t}_n \mid \Delta t(\Theta; z_n)\Big) \int dd_L\, p\Big(\widehat{d}_{L,n} \mid d_L, \iota, \mathcal{M}\Big)\, p(d_L \mid \Theta; z_n), \tag{14.8}$$

where $\Delta t(\Theta; z)$ encodes the integrated propagation speed correction

$$\Delta t(z) \;\simeq\; \int_0^z \frac{dz'}{H(z')} \frac{\delta c_T(z';\Theta)}{1+\delta c_T(z';\Theta)}\,, \tag{14.9}$$

and $p(d_L|\Theta; z)$ incorporates amplitude damping modifications (e.g. friction parameter $\nu(z)$) induced by the curvature sector.

Priors and hierarchical nuisance structure

We adopt a factorized prior $\pi(\Theta,\psi) = \pi_{\rm gates}(\Theta)\, \pi_{\rm astro}(\Theta)\, \pi_{\rm nuis}(\psi)$, where $\pi_{\rm gates}$ implements: *(G1)* no ghosts ($\mathsf{K} \succ 0$), *(G2)* subluminality ($0 < c_s(\theta), c_T \le 1$), *(G3)* positive entropy production ($\eta \ge 0$), *(G4)* restoration to GR–like propagation post–inflation, and *(G5)* observational viability (e.g. BBN). Nuisance parameters carry Gaussian or weakly informative hyperpriors, with calibration hierarchies (e.g. bandpowers share beam calibration hyperparameters).

Gradients, compression, and Fisher preconditioning

Let $\mathbf{J} \equiv \partial \mathbf{m}/\partial\Theta$ evaluated at a reference point $\Theta_\star$. Score compression yields a near–lossless summary

$$\mathbf{t} = \mathbf{J}_\star^T \mathbf{C}_\star^{-1} (\mathbf{d} - \mathbf{m}_\star), \qquad \mathcal{I}_\star = \mathbf{J}_\star^T \mathbf{C}_\star^{-1} \mathbf{J}_\star \quad \text{(Fisher)}, \tag{14.10}$$

useful for proposal preconditioning in HMC/NUTS or as a diagnostic. Automatic differentiation through the modified Boltzmann and transfer solvers (App. VIII) supplies $\mathbf{J}$ stably.

Masks, mixing, and E/B leakage

Pseudo-C_ℓ estimators relate full–sky spectra to masked ones via

$$\widehat{\mathbf{C}} = \mathbf{M}\,\mathbf{C} + \mathbf{n}, \qquad \mathbf{M}_{\ell\ell'}^{XY} = \frac{2\ell'+1}{4\pi} \sum_L (2L+1)\, W_L^{XY} \begin{pmatrix} \ell & \ell' & L \\ 0 & 0 & 0 \end{pmatrix}^2, \tag{14.11}$$

where W_L^{XY} are mask spectra and the $3j$ symbol captures mode coupling. For polarization, we use pure–E/B mixing kernels to control leakage; covariance is propagated accordingly.

Putting it together: posterior and evidence

The posterior is

$$p(\Theta, \psi \mid \mathbf{d}) \;\propto\; \mathcal{L}_{\rm joint}(\mathbf{d} \mid \Theta, \psi)\; \pi(\Theta, \psi), \tag{14.12}$$

sampled with HMC/nested methods. Model selection versus ΛCDM uses Bayes factors $\mathcal{B} = \mathcal{Z}_{\rm spec}/\mathcal{Z}_{\Lambda\rm CDM}$, with thermodynamic integration or importance nested sampling; information criteria (DIC/WAIC) serve as cross–checks under matched priors.

Consistency gates as on–the–fly checks. At each proposal, we verify numerically: (i) hyperbolicity cones inside $g_{\rm eff}$; (ii) c_T bounds from multi–messenger catalogs; (iii) BBN consistency of $N_{\rm eff}$ given reheating (Vol. II, Sec. 6.2); (iv) small anisotropy bounds for $c_s(\theta)$ in the primordial sector.

Outcome. Eqs. (14.3)–(14.9) define a modular, differentiable likelihood that fuses early– and late–time information with GW propagation, enabling coherent constraints on $\Theta_{\rm spec}$ under mathematically explicit priors induced by the theory's gates.

14.2 Tension Diagnostics: Internal Consistency and Cross-Checks

Objective. Having defined the joint likelihood (Sec. 14.1), we require quantitative tools to (i) diagnose *internal* consistency among data subsets and (ii) assess *external* concordance with legacy baselines (e.g. ΛCDM posteriors). We present a battery of *tension diagnostics* that are likelihood–aware, robust to mild non–Gaussianity, and compatible with the theory–driven priors (gates) of spectral cosmology.

Posterior Predictive Checks (PPC)

Given a fitted posterior $p(\Theta, \psi \mid \mathbf{d})$, the posterior predictive distribution (PPD) for a statistic $T(\mathbf{d})$ is

$$p(T^* \mid \mathbf{d}) \;=\; \int d\Theta\, d\psi\; p(T^* \mid \Theta, \psi)\;\; p(\Theta, \psi \mid \mathbf{d}), \qquad p(T^* \mid \Theta, \psi) = \int d\mathbf{d}^*\, \delta\big(T^* - T(\mathbf{d}^*)\big)\, p(\mathbf{d}^* \mid \Theta, \psi). \tag{14.13}$$

Define a PPC p–value:

$$p_{\rm PPC} \;=\; \Pr[T(\mathbf{d}^*) \geq T(\mathbf{d}) \mid \mathbf{d}]\,, \tag{14.14}$$

computed by simulating summaries $\mathbf{d}^*$ at posterior draws and passing them through the same mask/estimator pipeline. Values $p_{\rm PPC} \ll 0.01$ (or $\gg 0.99$) flag mis–modeling or outliers. We design T to target specific residuals (e.g. high–ℓ EE spectrum curvature, scale–dependent galaxy bias, WL E/B leakage).

Split Statistics and Parameter–Shift Measures

Split the data into disjoint blocks $\mathcal{D}_1, \mathcal{D}_2$ (e.g. low/high ℓ, redshift tomes, hemispheres). Let $\bar{\Theta}_i$ be posterior means and $\mathbf{C}_i$ their covariances from each split. The *parameter shift* statistic reads

$$\Delta\chi^2_{\rm par} \;=\; (\bar{\Theta}_1 - \bar{\Theta}_2)^T\, (\mathbf{C}_1 + \mathbf{C}_2)^{-1}\, (\bar{\Theta}_1 - \bar{\Theta}_2), \qquad P_{\rm shift} = \Pr\big(\chi^2_k \geq \Delta\chi^2_{\rm par}\big)\,, \tag{14.15}$$

with $k = \dim(\Theta_{\rm eff})$ after compression to an informative subspace (e.g. via Fisher eigenmodes). Small $P_{\rm shift}$ indicates a tension between splits beyond expected statistical fluctuations.

Suspiciousness and Evidence Ratio Diagnostics

For two experiments A, B with posteriors $p_A(\Theta) \propto \mathcal{L}_A \pi$ and $p_B(\Theta) \propto \mathcal{L}_B \pi$, define evidences $\mathcal{Z}_A, \mathcal{Z}_B, \mathcal{Z}_{AB}$ under a common prior $\pi(\Theta)$. The *suspiciousness* statistic (a prior–volume–mitigated concordance measure) is

$$\mathcal{S} \;\equiv\; \frac{\mathcal{Z}_{AB}}{\mathcal{Z}_A\, \mathcal{Z}_B}\, \frac{\int d\Theta\; \pi(\Theta)^2}{\int d\Theta\; \pi(\Theta)}\,, \qquad \ln \mathcal{S} \approx -\tfrac{1}{2}\, \Delta\boldsymbol{\mu}^T\, \mathbf{C}^{-1} \Delta\boldsymbol{\mu} \quad \text{(Gaussian limit)}, \tag{14.16}$$

where $\Delta\boldsymbol{\mu}$ is the difference of posterior means and $\mathbf{C}$ the sum of covariances. Negative $\ln\mathcal{S}$ at $|\ln\mathcal{S}| \gtrsim \mathcal{O}(5)$ indicates notable discordance. We compute $\mathcal{Z}$ with nested sampling (thermodynamic integration as cross–check), respecting gate priors.

Goodness–of–Fit Decomposition and χ^2 Budget

The global deviance decomposes into block contributions:

$$-2\ln\mathcal{L}_{\text{joint}} = \sum_b -2\ln\mathcal{L}_b + \Delta_{\text{cross}}\,, \qquad b \in \{\text{TT}, \text{TE}, \text{EE}, \kappa\kappa, \kappa g, gg,\ P,\ f\sigma_8,\ \text{GW}\}, \tag{14.17}$$

with Δ_{cross} capturing cross–covariances (Sec. 14.1). At the posterior mode, we tabulate per–block residuals against effective degrees of freedom, using Monte Carlo noise realizations to calibrate the expected distribution under the fitted model. Outliers localize model–data mismatches.

Profile and Posterior Predictive Distributions for Key Tensions

We implement dedicated profiles for well–known tensions:

$$H_0 \text{ profile:} \quad \mathcal{L}_{\text{prof}}(H_0) = \max_{\Theta\setminus H_0,\psi} \mathcal{L}_{\text{joint}}(\mathbf{d} \mid \Theta, \psi)\,\pi(\Theta, \psi), \tag{14.18}$$

$$S_8 \text{ profile:} \quad \mathcal{L}_{\text{prof}}(S_8) = \max_{\Theta\setminus S_8,\psi} \mathcal{L}_{\text{joint}}(\cdots). \tag{14.19}$$

Comparing $\mathcal{L}_{\text{prof}}$ with independent H_0 and S_8 posteriors yields calibrated tension levels (in σ or via Bayesian model comparison with nuisance hyperparameters that emulate systematic shifts).

Internal Consistency with Gates (S/G/TQ/D)

Each posterior draw is vetted by the gates; we record *gate efficiency* and *gate pressure*:

$$\epsilon_{\text{gate}} = \frac{N_{\text{accepted}}}{N_{\text{proposed}}}, \qquad \Pi_{\text{gate}}(\Theta) = \big(\lambda_{\min}(\mathsf{K}),\ \min c_s^2,\ \min c_T^2,\ \sigma_{\text{diss}}\big), \tag{14.20}$$

where small Π_{gate} components indicate proximity to a boundary (e.g. $c_T \to 1^-$). Mapping tension directions in (H_0, S_8) to gate–pressure gradients shows whether data are pushing the theory toward unstable or disallowed regions, an *actionable* diagnostic distinguishing statistical flukes from structural inadequacy.

Anisotropy and Hemispherical Splits

For spectral anisotropy parameters $\mathcal{A}_{\text{aniso}}, \hat{\chi}^{(0)}$, we form hemispherical or multipole–parity splits. Let $\mathbf{d}_N, \mathbf{d}_S$ be northern/southern sky summaries; compute $\Delta\chi^2_{\text{par}}$ as in Eq. (14.15). Additionally, define an *angular drift* statistic for the preferred direction:

$$\mathcal{D}_\chi \equiv \arccos\big(\hat{\chi}^{(0)}_{\text{CMB}} \cdot \hat{\chi}^{(0)}_{\text{LSS}}\big), \tag{14.21}$$

with its PPD obtained by simulating skies under the posterior. Significant alignment (or misalignment) beyond PPD expectations is a targeted signature of the spectral director field.

Non-Gaussian Summary Residuals

For LSS bispectrum (or Minkowski functionals in WL maps), let $\widehat{B}_\alpha$ be binned configurations with theory $B_\alpha(\Theta)$. Define the normalized residual vector $\mathbf{r} = \mathbf{C}_B^{-1/2}(\widehat{B} - B)$ and use its Mahalanobis norm $\chi^2_B = \mathbf{r}^T\mathbf{r}$ as a goodness metric, with $\mathbf{C}_B$ estimated by simulations or response theory. Departures concentrated in *squeezed* shapes would point to $c_s(\theta)$–induced mode coupling rather than foregrounds.

Decision Table and Escalation Criteria

We consolidate diagnostics into a decision table:

1. If $p_{\mathrm{PPC}} \in [0.05, 0.95]$ for all primary $T(\mathbf{d})$, $P_{\mathrm{shift}} > 0.05$ for key splits, and $\ln \mathcal{S} > -2$, the dataset is *globally consistent.*
2. If one of $\{p_{\mathrm{PPC}}, P_{\mathrm{shift}}, \ln \mathcal{S}\}$ fails mildly, we expand nuisance models (ψ) and repeat; if persistent, promote to *systematics audit.*
3. If multiple diagnostics fail and gate pressure is high, we consider *model extensions* within spectral cosmology (e.g. alternative V_η, defect sub–sector refinements) before appealing to external additions.

Outcome. Section 14.2 defines a practical, quantitative *tension dashboard.* Its outputs (PPC values, parameter shifts, suspiciousness, gate pressures) guide whether tensions reflect noise/systematics, point to specific phenomenological refinements, or constitute evidence against the current spectral–cosmology instantiation.

14.3 Drivers of the H_0 and S_8 Tensions in Spectral Cosmology

Objective. We identify which *spectral* parameters primarily control the classical cosmological tensions—the CMB–inferred Hubble constant H_0 and the low-redshift clustering amplitude $S_8 \equiv \sigma_8(\Omega_m/0.3)^{1/2}$—and construct *controlled deformations* that can relieve them while respecting the gates (S/G/TQ/D) and external bounds (BBN, ΔN_{eff}, GW speeds).

Analytic sensitivities: H_0 via r_s and S_8 via growth

The CMB–inferred H_0 is controlled by the comoving sound horizon at baryon drag,

$$r_s(z_d) = \int_{z_d}^{\infty} \frac{c_s(z)}{H(z)}\, dz, \qquad c_s(z) = \frac{1}{\sqrt{3\,(1+R_b(z))}}, \quad R_b(z) = \frac{3\rho_b}{4\rho_\gamma}. \tag{14.22}$$

At fixed acoustic scale $\theta_* = r_s/D_A(z_*)$, a fractional change $\delta \ln r_s$ induces

$$\delta \ln H_0 \;\simeq\; -\,\delta \ln r_s \;+\; \alpha_{\mathrm{geo}}\, \delta \ln \Omega_m \;+\; \alpha_{\mathrm{curv}}\, \delta \Omega_k \;+\; \cdots\,, \tag{14.23}$$

with small geometric coefficients $\alpha_{\mathrm{geo}}, \alpha_{\mathrm{curv}}$. In spectral cosmology, $H(z)$ and $c_s(z)$ depend on *early* fractional energy in the viscosity sector,

$$f_\eta(z) \;\equiv\; \frac{\rho_\eta(z)}{\rho_{\mathrm{tot}}(z)}\Big|_{\text{pre recomb}}, \qquad \delta \ln r_s \;\approx\; -\frac{1}{2}\int_{z_d}^{\infty} \frac{f_\eta(z)}{1+z}\,\frac{dz}{\mathcal{E}(z)} \;+\; \frac{1}{2}\,\overline{\delta \ln c_s}, \quad \mathcal{E}(z) \equiv \frac{H(z)}{H_0}, \tag{14.24}$$

where the $\overline{\delta \ln c_s}$ term (baryon–photon physics) remains small under our gates. Thus *early* η–coherence ($f_\eta > 0$ at $z \sim 10^3$) can shrink r_s, raising the inferred H_0.

For S_8, the linear growth $D(a)$ obeys

$$D'' + \left(2 + \frac{H'}{H}\right) D' - \frac{3}{2}\,\mu(a,k)\,\Omega_m(a)\, D = 0, \qquad \mu(a,k) \equiv \frac{G_{\mathrm{eff}}(a,k)}{G_N}, \tag{14.25}$$

with primes $' = d/d\ln a$. Spectral curvature terms and residual η–relaxation enter via a modified Poisson sector $\mu(a,k) = 1 + \Delta\mu_\eta(a,k)$ and a mild gravitational slip $\gamma(a,k) \equiv \Phi/\Psi = 1 + \Delta\gamma_\eta$. To leading order,

$$\delta \ln S_8 \;\simeq\; \frac{1}{2}\,\delta \ln \sigma_8 \;\approx\; -\frac{3}{5}\int_{a_{\mathrm{LSS}}}^{1} \Delta\mu_\eta(a,k_*)\,\Omega_m(a)\, d\ln a \;+\; \beta_{\mathrm{EDE}}\, f_\eta^{\mathrm{early}} \;+\; \cdots\,, \tag{14.26}$$

where $k_* \sim 0.2\, h\,\mathrm{Mpc}^{-1}$ sets the σ_8 scale and β_{EDE} captures indirect changes in transfer functions from early η. Late–time $\Delta\mu_\eta < 0$ (effective weakening of clustering) reduces S_8.

Spectral knobs and their principal directions

We summarize the *minimal* deformation set and their dominant effects:

(K1) Early η fraction: $f_\eta^{\rm early}(z) \equiv \rho_\eta/\rho_{\rm tot}$ for $z \gtrsim 10^3$. $\Rightarrow \downarrow r_s \;\Rightarrow \uparrow H_0$, BBN/$N_{\rm eff}$ gate.
(K2) Late η relaxation: $\delta(a)$ in $w_{\rm eff}(a) = -1 + \delta(a)$, $z \lesssim 2$. $\Rightarrow$ background distances & growth damping (via $\Delta\mu$
(K3) Curvature response: $\Delta\mu_\eta(a,k)$, $\Delta\gamma_\eta(a,k)$. $\Rightarrow \downarrow S_8$ if $\Delta\mu_\eta < 0$ at $z \lesssim 1$.
(K4) Defect budget: $\Omega_{\rm def} \equiv \rho_{\rm defects}/\rho_{\rm crit}$. $\Rightarrow$ broad–band power tilt, small non–Gaussian tails;
(K5) Mild anisotropy: g_*, $\hat{\chi}^{(0)}$. $\Rightarrow$ angular systematics cross–checks; kept $\ll$ CMB l

(14.27)

Linearizing observables around a reference spectral point $\Theta_{\rm ref}$,

$$\begin{pmatrix} \delta \ln H_0 \\ \delta \ln S_8 \end{pmatrix} \simeq \underbrace{\begin{pmatrix} \partial \ln H_0/\partial f_\eta^{\rm early} & \partial \ln H_0/\partial \langle\delta\rangle & \partial \ln H_0/\partial \mu_0 \\ \partial \ln S_8/\partial f_\eta^{\rm early} & \partial \ln S_8/\partial \langle\delta\rangle & \partial \ln S_8/\partial \mu_0 \end{pmatrix}}_{\equiv\, \mathbf{J}} \begin{pmatrix} \delta f_\eta^{\rm early} \\ \delta\langle\delta\rangle \\ \delta\mu_0 \end{pmatrix}, \qquad \mu_0 \equiv \Delta\mu_\eta(z{\sim}0.5, k_*), \tag{14.28}$$

reveals the principal directions: H_0 responds steeply to $f_\eta^{\rm early}$, while S_8 responds mostly to μ_0 and, to a lesser degree, to $\langle\delta\rangle \equiv \int \delta(a)\, d\ln a$.

Controlled deformations and priors (gates)

We implement *tempered* deformations:

$$\begin{aligned} &\text{(D1) Early window:} \quad f_\eta^{\rm early}(z) = f_*\, \mathcal{W}(z; z_c, \Delta z)\,, \quad \mathcal{W} \text{ narrow around recombination;} \\ &\text{(D2) Late drift:} \quad \delta(a) = \delta_0 \frac{1}{1+(a/a_t)^p}\,, \quad p \geq 2; \\ &\text{(D3) Growth response:} \quad \mu(a,k) = 1 + \mu_0 \frac{a^q}{1+(k/k_\mu)^n}\,, \end{aligned} \tag{14.29}$$

with *hard* priors from gates:

$$\textbf{BBN:}\ f_\eta^{\rm early} \lesssim \mathcal{O}(0.05), \quad \textbf{CMB:}\ |g_*| \ll 10^{-2}, \quad \textbf{GW:}\ |c_T - 1| \lesssim 10^{-15}\ \text{(today)}, \quad \textbf{Positivity:}\ \sigma_{\rm diss} \geq 0,\ \mathsf{K} \succ 0. \tag{14.30}$$

Soft priors enforce smoothness (p, q, n not too small) and avoid sharp features incompatible with linear theory validity.

Two illustrative relief patterns

EDE–like spectral relief. Choose $(f_*, \delta_0, \mu_0) = (0.03, 0, 0)$ with narrow $\mathcal{W}$ at $z{\sim}3000$:

$$\delta \ln r_s \simeq -\mathcal{O}(1\%) \;\Rightarrow\; \delta H_0/H_0 \simeq +\mathcal{O}(1\text{–}2\%)\,, \quad \delta S_8 \text{ small.}$$

This can lift H_0 modestly but typically leaves the S_8 tension unchanged. Gate checks: BBN and CMB damping tail.

Balanced spectral relief. Combine $(f_*, \delta_0, \mu_0) = (0.02, 0.02, -0.05)$ with $a_t{\sim}0.5$, $k_\mu{\sim}0.2\,h\,{\rm Mpc}^{-1}$:

$$\delta H_0/H_0 \simeq +1\% \text{ from } f_*, \qquad \delta \ln S_8 \simeq -2\% \text{ from } \mu_0 \text{ and mild } \delta(a).$$

This pattern addresses both tensions concurrently with small, gate–compatible deformations. Robustness requires passing PPCs (Sec. 14.2) and the joint-likelihood stress tests (Sec. 14.1).

Degeneracies, null tests, and failure modes

We track degeneracies between $(f_*, \omega_b, \omega_c)$ in r_s and between $(\mu_0, \sigma_8^{\rm ini})$ in S_8 using Fisher eigenmodes and profile likelihoods. *Null tests:*

- Early f_* raises H_0 but *also* shifts diffusion damping; Planck/ACT high-ℓ PPC must remain acceptable.
- Late $\mu_0 < 0$ lowers S_8 but must preserve RSD and weak-lensing scale–dependence within current systematics.
- $\delta(a)$ alters distances; BAO+SNe residuals provide a sensitive veto against over–fitting H_0.

Failure modes include gate violations (superluminal c_T, negative kinetic directions), or evidence of significant *directional* anisotropy in LSS beyond our allowed g_*.

Decision outcomes

We define three outcomes after full sampling with deformations (14.29):

1. **Pass (balanced relief):** Tensions reduced to $\lesssim 2\sigma$ with gates satisfied and $\Delta \ln \mathcal{Z}$ not disfavouring the model. This favours the balanced spectral relief pattern.
2. **Partial pass ($\mathbf{H}_0$ only):** EDE–like relief raises H_0 but leaves S_8; proceed to refine $\mu(a,k)$ calibration from the curvature sector (Vol. I, Ch. 3; Vol. II, Ch. 8).
3. **Fail:** PPCs or gate pressures indicate structural inadequacy; escalate to extended curvature operators or revisit η potential (see Vol. II, Part IV).

Summary. Early η–energy chiefly controls r_s (hence H_0), while late–time curvature response $\mu(a,k)$ and mild η–relaxation shape S_8. A *balanced* spectral deformation can alleviate both tensions within gate constraints, providing a concrete, falsifiable alternative to ΛCDM adjustments.

14.4 Designing Decisive Tests and Targeted Nulls

Objective. We construct *targeted, falsifiable* tests that isolate the distinctive mechanisms of spectral cosmology—early η energy, late η relaxation, and curvature response $\mu(a,k)$—while remaining orthogonal to common systematics. Each test is expressed as a statistic with a predicted scaling, a null expectation under ΛCDM, and gate–compatibility (S/G/TQ/D).

Principles of test design

(i) *Orthogonality:* choose summary statistics minimally degenerate with $\{\omega_b, \omega_c, n_s, A_s, \tau\}$. (ii) *Localization:* window functions in redshift/scale to isolate early ($z \sim 10^3$) vs. late ($z \lesssim 1$) physics. (iii) *Nulls:* construct signed quantities with ΛCDM mean zero (up to known, calibratable leakage). (iv) *Joint evidence:* report both χ^2-improvement and Bayes factors $\Delta \ln \mathcal{Z}$ with PPCs.

CMB polarization TB/EB nulls and damping-tail lever arm

Spectral anisotropy (g_*) and early η fraction ($f_\eta^{\rm early}$) leave correlated imprints in *(a)* the polarization-rotation–like TB/EB spectra and *(b)* the high-ℓ damping tail.

Null statistic. Define $\widehat{\Delta}_{\rm pol}$ via inverse–covariance filtering

$$\widehat{\Delta}_{\rm pol} \equiv \sum_{\ell=\ell_{\rm min}}^{\ell_{\rm max}} \mathbf{d}_\ell^\top \mathbf{C}_\ell^{-1} \begin{pmatrix} C_\ell^{TB} \\ C_\ell^{EB} \\ \Delta C_\ell^{TT,\rm damp} \end{pmatrix}_{\rm model}, \qquad \mathbf{d}_\ell \equiv \begin{pmatrix} \widehat{C}_\ell^{TB} \\ \widehat{C}_\ell^{EB} \\ \widehat{C}_\ell^{TT} - C_\ell^{TT,\Lambda\rm CDM} \end{pmatrix}, \tag{14.31}$$

with $\Delta C_\ell^{TT,\rm damp}$ a template localized at $\ell \gtrsim 1200$. Under ΛCDM+Λ parity, $\mathbb{E}[\widehat{\Delta}_{\rm pol}] = 0$; spectral cosmology predicts

$$\mathbb{E}[\widehat{\Delta}_{\rm pol}] \;\propto\; \alpha_1\, g_* \;+\; \alpha_2\, f_\eta^{\rm early}, \qquad |\alpha_{1,2}| = \mathcal{O}(1) \text{ (experiment–dependent)}. \tag{14.32}$$

Gate constraints require $|g_*| \ll 10^{-2}$ and PPC agreement on the damping tail (Sec. 14.2).

GW propagation triad: speed, friction, and lensing slip

The emergent curvature sector implies (small) deviations in the GW propagation equation,

$$h'' + (2+\nu(a))\,\mathcal{H}\,h' + c_T^2 k^2 h = \Pi_{\rm src}, \qquad \nu(a) \equiv -\frac{d\ln M_{\rm eff}}{d\ln a}, \tag{14.33}$$

with $|c_T - 1| \ll 1$ today (multi–messenger gate) and a *friction drift* $\nu(a)$ at $z \lesssim 1$ if $M_{\rm eff}$ relaxes mildly.

Distance–duality test. Comparing standard siren distances $D_L^{\rm GW}$ to EM distances $D_L^{\rm EM}$ in the same redshift bins,

$$\mathcal{R}_{\rm GW/EM}(z) \;\equiv\; \frac{D_L^{\rm GW}(z)}{D_L^{\rm EM}(z)} \;=\; \exp\!\left[\frac{1}{2}\int_0^z \frac{\nu(z')}{1+z'}\,dz'\right] \;\overset{\Lambda\rm CDM}{=}\; 1. \tag{14.34}$$

A non–unity $\mathcal{R}_{\rm GW/EM}$ at $z \lesssim 1$ is a clean late–time signature of spectral relaxation; bounds on c_T ensure the *phase* propagation is luminal, isolating the *amplitude* channel.

LSS anisotropy and growth–response null

The late–time curvature response modifies the Poisson sector via $\mu(a,k)$ and can carry a weak directional imprint aligned with $\hat{\chi}^{(0)}$ (bounded by CMB).

Quadrupolar power null. Construct the pseudo–C_ℓ quadrupole of the galaxy power spectrum in tomographic bins,

$$Q(k,z) \;\equiv\; \frac{\int d\Omega_{\hat{k}}\; P_g(\mathbf{k},z)\,\mathcal{P}_2(\hat{k}\cdot\hat{\chi}^{(0)})}{\int d\Omega_{\hat{k}}\; P_g(\mathbf{k},z)}, \qquad \overset{\Lambda\rm CDM}{=}\; 0 \;\; \text{(after RSD removal)}, \tag{14.35}$$

with RSD anisotropy subtracted using multipole fits. Spectral prediction:

$$Q(k,z) \;\simeq\; \beta_Q\,\Delta\mu_\eta(a,k)\,\left(\hat{\chi}^{(0)}\ \text{align.}\right), \qquad |\beta_Q| = \mathcal{O}(1). \tag{14.36}$$

Consistency requires $|Q| \ll 10^{-2}$ in CMB–calibrated early–anisotropy models, yet $|Q| \sim$ few $\times\, 10^{-3}$ remains testable with DESI/Euclid.

Growth–response ratio. Define a scale–selective estimator

$$\mathcal{G}(z) \;\equiv\; \frac{f\sigma_8(k_{\rm lo},z)}{f\sigma_8(k_{\rm hi},z)}, \qquad \overset{\Lambda\rm CDM}{=}\; 1 \pm \epsilon_{\rm sys}, \tag{14.37}$$

with $(k_{\rm lo}, k_{\rm hi})$ chosen straddling k_μ where $\mu(a,k)$ transitions. A significant, coherent deviation of $\mathcal{G}(z)$ from unity across bins signals scale–dependent $\Delta\mu_\eta$.

CMB×LSS×lensing: ISW–lensing bispectrum null

Late–time potential evolution induced by $\delta(a)$ (Sec. 14.1) generates an ISW–lensing bispectrum

$$B^{T\phi\phi}_{\ell_1\ell_2\ell_3} \;\propto\; \int dz\; \frac{d}{dz}[\Phi(z)+\Psi(z)]\;\; W_\phi^2(z)\,, \tag{14.38}$$

with ΛCDM expectation fixed by background distances. Spectral cosmology predicts a fractional shift

$$\frac{\Delta B^{T\phi\phi}}{B^{T\phi\phi}_{\Lambda\rm CDM}} \;\simeq\; \gamma_{\rm ISW}\,\langle\delta\rangle \;+\; \gamma_\mu\,\overline{\Delta\mu_\eta}, \tag{14.39}$$

providing a *joint* null that correlates the same parameters affecting S_8.

Decision metrics and forecasted power

For each null $X \in \{\widehat{\Delta}_{\rm pol},\ \mathcal{R}_{\rm GW/EM},\ Q,\ \mathcal{G},\ \Delta B^{T\phi\phi}\}$ we define

$$\mathrm{SNR}_X^2 \ \equiv\ \sum_{\alpha,\beta} X_\alpha\ [\mathrm{Cov}(X)]^{-1}_{\alpha\beta}\ X_\beta, \qquad \Delta \ln \mathcal{Z}\ \approx\ \frac{1}{2}\,\mathrm{SNR}^2_{\rm joint}\ -\ \Delta k\ \ln V, \tag{14.40}$$

where Δk is the effective parameter penalization and V the Occam volume ratio. *Pass* requires (i) $\mathrm{SNR}_{\rm joint} \gtrsim 3$ in favour of the spectral model and (ii) all gates satisfied within priors (Sec. 12.1).

Failure modes and vetoes

- **GW veto:** any $|c_T - 1|$ beyond 10^{-15} today or $|\mathcal{R}_{\rm GW/EM} - 1|$ inconsistent with $\nu(a)$ priors triggers a curvature–sector failure.
- **Anisotropy veto:** $|Q|$ or $|g_*|$ exceeding CMB bounds invalidates director–aligned deformations.
- **PPC veto:** high–ℓ CMB PPC residuals consistent with $f_\eta^{\rm early} = 0$ exclude EDE–like relief.

Summary. The suite $\{\widehat{\Delta}_{\rm pol},\ \mathcal{R}_{\rm GW/EM},\ Q,\ \mathcal{G},\ \text{ISW–lensing bispectrum}\}$ forms an orthogonal, gate–aware battery of *decisive tests* for spectral cosmology. Each admits an analytic scaling with the deformation knobs $\{f_\eta^{\rm early},\ \delta(a),\ \mu(a,k)\}$, enabling transparent interpretation of positive or null results against ΛCDM.

Chapter 15

Forecasts and Discovery Space

15.1 Fisher/Simulation Forecasts for $(n_s, r, f_{\rm NL}, g_*, c_T)$

Scope and parameter set

We forecast joint constraints on the primordial and propagation parameters

$$\boldsymbol{\theta} \;=\; \left\{ n_s,\; r,\; f_{\rm NL},\; g_*,\; c_T \right\},$$

with the following operational definitions:

- n_s: tilt of the scalar curvature spectrum $\mathcal{P}_\zeta(k) \propto k^{\,n_s-1}$ at a pivot $k_\star$.
- r: tensor-to-scalar ratio at $k_\star$, $r \equiv \mathcal{P}_T/\mathcal{P}_\zeta$.
- $f_{\rm NL}$: amplitude of primordial non-Gaussianity (local shape for large-scale structure forecasts unless noted).
- g_*: quadrupolar statistical anisotropy in the primordial spectrum, $\mathcal{P}_\zeta(\mathbf{k}) = \mathcal{P}_0(k)\,[1 + g_* P_2(\mu)]$, $\mu \equiv \hat{\mathbf{k}}\cdot\hat{\chi}^{(0)}$.
- c_T: tensor-wave speed during propagation (late-time), entering CMB B-mode transfer functions only weakly but tightly constrained by multi-messenger timing.

Unless stated otherwise, all power spectra are evaluated in the *emergent* background of Secs. **??** and **??**, with $M_{\rm eff}$, slow parameters, and sound speeds inherited from the spectral aether.

CMB Fisher information (T, E, B, lensing)

For a survey with sky fraction $f_{\rm sky}$, multipole range $\ell_{\rm min} \le \ell \le \ell_{\rm max}$, and noise spectra $N_\ell^{TT}, N_\ell^{EE}, N_\ell^{BB}$, the Gaussian Fisher matrix for parameters θ_i is

$$F_{ij}^{\rm CMB} = \sum_{\ell=\ell_{\rm min}}^{\ell_{\rm max}} \frac{2\ell+1}{2}\, f_{\rm sky}\, {\rm Tr}\big[\mathbf{C}_\ell^{-1}\,\partial_i \mathbf{C}_\ell\,\mathbf{C}_\ell^{-1}\,\partial_j \mathbf{C}_\ell\big]\,, \qquad \mathbf{C}_\ell = \begin{pmatrix} C_\ell^{TT}+N_\ell^{TT} & C_\ell^{TE} & 0 \\ C_\ell^{TE} & C_\ell^{EE}+N_\ell^{EE} & 0 \\ 0 & 0 & C_\ell^{BB}+N_\ell^{BB} \end{pmatrix}. \tag{15.1}$$

We include lensing B-modes as residual noise after delensing: $N_\ell^{BB} \to N_\ell^{BB} + \alpha_{\rm dl}^2\, C_\ell^{BB,{\rm lens}}$, with $\alpha_{\rm dl} \in [0,1]$ the *map-level* delensing factor. The derivatives $\partial_i \mathbf{C}_\ell$ are computed with respect to $(n_s, r, f_{\rm NL}, g_*, c_T)$ via the transfer functions:

- r and c_T: $C_\ell^{BB,{\rm tens}} \propto r\, T_\ell^{(B)}(c_T)^2$; for $|c_T - 1| \ll 1$, we linearize $T_\ell^{(B)}$ in δc_T.
- n_s: $\partial_{n_s} \ln C_\ell^{XY} \simeq \partial_{n_s} \ln \mathcal{P}_\zeta(k)$ evaluated near $k \simeq \ell/\chi_*$.
- $f_{\rm NL}$: enters CMB primarily via ISW–lensing and higher-order statistics; for two-point forecasts we set $\partial_{f_{\rm NL}} C_\ell \approx 0$ and defer its weight to LSS/bispectrum blocks.

- g_*: produces off-diagonal covariance and m-dependent spectra; at leading order, it induces a *quadrupolar* modulation $\Delta C_{\ell m,\ell' m'}$. For Fisher-level sensitivity, we adopt the quadratic-estimator compression equivalent to an effective variance $\sigma^2(g_*)^{-1} \sim \sum_{\ell m}(\mathrm{SNR}^{(2)}_{\ell m})^2$.

We report $\sigma(\theta_i) = \sqrt{(F^{-1})_{ii}}$ and correlation coefficients $\rho_{ij} = (F^{-1})_{ij}/[\sigma(\theta_i)\sigma(\theta_j)]$. Delensing ($\alpha_{\rm dl}\downarrow$) and low foregrounds shrink $\sigma(r)$ dramatically; $\sigma(c_T)$ is dominated by the GW timing block below.

Galaxy/LSS Fisher: power spectrum and scale-dependent bias

For a tomographic bin at redshift z with comoving volume $V(z)$, number density $n_g(z)$, bias $b_1(z)$, and redshift-space distortions (RSD), the galaxy power spectrum reads

$$P_g(k,\mu;z) = \left[b_1(z) + \Delta b_{f_{\rm NL}}(k,z) + f(z)\mu^2\right]^2 P_m(k,z)\left[1 + g_* P_2(\mu)\right], \tag{15.2}$$

with growth rate $f(z)$, $\mu \equiv \hat{\mathbf{k}}\cdot\hat{\mathbf{n}}$, and the local-shape scale-dependent bias

$$\Delta b_{f_{\rm NL}}(k,z) = 2\, f_{\rm NL}\left[b_1(z) - 1\right]\delta_c\, \frac{3\Omega_m H_0^2}{k^2 T(k)\, D(z)}\,, \tag{15.3}$$

($\delta_c \simeq 1.686$, transfer function $T(k)$, growth $D(z)$). The (Gaussian) Fisher per bin is

$$F^{\rm LSS}_{ij}(z) = \frac{V(z)}{8\pi^2}\int_{k_{\rm min}}^{k_{\rm max}} k^2\, dk \int_{-1}^{1} d\mu\, \frac{\partial_i \ln P_g\, \partial_j \ln P_g}{\left[1 + \frac{1}{n_g P_g}\right]^2}\,, \tag{15.4}$$

with $\partial_i \ln P_g$ taken with respect to $\{n_s, f_{\rm NL}, g_*\}$ (and nuisance parameters marginalized). The g_*-derivative is $\partial_{g_*}\ln P_g = P_2(\mu)/[1 + g_* P_2(\mu)] \approx P_2(\mu)$ at small g_*, highlighting the importance of wide angular coverage and precise RSD modeling. We sum over redshift bins and tracers (ELGs, LRGs, quasars) and impose $k_{\rm max}(z)$ from nonlinearity (e.g. $\sigma^2(R_{\rm NL}, z) = 0.25$).

Primordial bispectrum and LSS multi-tracer

Robust $f_{\rm NL}$ forecasts require (i) CMB bispectrum and/or (ii) LSS multi-tracer techniques to beat cosmic variance:

$$F^{\rm MT}_{f_{\rm NL} f_{\rm NL}} \sim \int \frac{d^3k}{(2\pi)^3}\, \frac{\left[\sum_a w_a \partial_{f_{\rm NL}} \delta_a(\mathbf{k})\right]^2}{\sum_{ab} w_a w_b\, {\rm Cov}[\delta_a, \delta_b]}\,, \tag{15.5}$$

where indices a, b run over tracers with distinct biases b_a, and optimal weights w_a minimize the covariance. Forecasts with two tracers of different bias can approach $\sigma(f_{\rm NL}) \lesssim 1$ if shot noise is low and large scales $k \lesssim 0.01\, h\,{\rm Mpc}^{-1}$ are accessible.

Multi-messenger constraints on c_T

For standard sirens with identified EM counterparts at redshifts z_n, the arrival-time residual between GWs (speed c_T) and light (speed c) is

$$\Delta t_n \simeq \int_0^{z_n} \frac{dz}{H(z)}\left(\frac{1}{c_T(z)} - \frac{1}{c}\right) \approx \frac{\delta_T}{c}\, D_{\rm com}(z_n)\,, \qquad \delta_T \equiv 1 - \frac{c_T}{c}\,, \tag{15.6}$$

for small, redshift-independent δ_T. The Fisher element is then

$$F^{\rm GW}_{c_T c_T} \simeq \sum_{n=1}^{N_{\rm siren}} \frac{1}{\sigma^2_{t,n}}\left(\frac{\partial \Delta t_n}{\partial c_T}\right)^2 = \sum_n \frac{D_{\rm com}(z_n)^2}{c^2\, \sigma^2_{t,n}}\,, \tag{15.7}$$

with timing uncertainty $\sigma_{t,n}$ including intrinsic lag modeling. PTA dispersion and high-frequency detectors add complementary bounds; together with CMB B-modes, this breaks r–c_T degeneracies.

Sensitivity maps and degeneracy breaking

To diagnose *where* the information comes from, we define per-mode contributions:

$$\mathcal{S}_i^{\rm CMB}(\ell) \equiv \frac{2\ell+1}{2}\, f_{\rm sky}\, {\rm Tr}\big[\mathbf{C}_\ell^{-1}\,\partial_i\mathbf{C}_\ell\,\mathbf{C}_\ell^{-1}\,\partial_i\mathbf{C}_\ell\big]^{1/2}, \quad \mathcal{S}_i^{\rm LSS}(k,\mu,z) \equiv \frac{1}{2}\left(\frac{V}{2\pi^2}\right)^{1/2} \frac{k\,|\partial_i \ln P_g|}{1+\frac{1}{n_g P_g}}. \tag{15.8}$$

These "heatmaps" guide survey optimization: e.g. large-scale modes $k\downarrow$ dominate $\mathcal{S}_{f_{\rm NL}}$; wide-μ coverage is critical for $\mathcal{S}_{g_*}$; deep delensing boosts $\mathcal{S}_r$. Jointly,

$$\mathbf{F}_{\rm tot} = \mathbf{F}^{\rm CMB} + \mathbf{F}^{\rm LSS} + \mathbf{F}^{\rm GW} + \mathbf{F}^{\rm (bispec)} + \cdots\,,$$

with nuisance/bias parameters profiled or marginalized. Degeneracies n_s–$f_{\rm NL}$ (tilt vs. scale-dependent bias) are broken by multi-redshift leverage; r–c_T by combining delensed B-modes with siren timing; g_* decorrelates from RSD quadrupoles through multi-angle analyses and survey rotation tests.

Simulation-based forecasts (beyond Fisher)

Non-Gaussian and anisotropic summaries motivate *likelihood-free* forecasts. We implement:

1. **Emulator pipelines:** Latin-hypercube sampling in $\boldsymbol{\theta}$, fast Boltzmann/LSS solvers, and Gaussian-process emulators for summary statistics $\mathbf{s}(\boldsymbol{\theta}) \in \{\mathcal{P}_\zeta,\, C_\ell^{XY},\, P_g,\, B_{\ell_1\ell_2\ell_3},\, {\rm MF}\}$.

2. **Simulation-Based Inference (SBI):** train neural density estimators $q_\phi(\boldsymbol{\theta}\,|\,\mathbf{s}_{\rm obs})$ on forward simulations with survey masks, noise, and systematics; validate coverage via simulation-based calibration.

3. **Hybrid Fisher–SBI:** use Fisher to set proposal covariances and compress summaries (e.g. score compression), then refine with SBI in degenerate directions (notably $f_{\rm NL}$, g_*).

This delivers robust posteriors when the Gaussian, diagonal-covariance assumptions of Eq. (15.1)–(15.4) break down.

Representative survey models (for reproducibility)

We will report baseline and optimistic forecasts under explicit assumptions, e.g.:

- **CMB-S4-like:** $f_{\rm sky} = 0.4$, noise $\Delta_T = 5\,\mu$K arcmin, $\Delta_P = 7\,\mu$K arcmin, beam $1.5'$, $\ell_{\rm max}^{T/E} = 3500$, $\ell_{\rm max}^{B} = 200$, delensing factor $\alpha_{\rm dl} = 0.2$.

- **Euclid/DESI-like LSS:** volumes $V \sim (10-50)\,h^{-3}{\rm Gpc}^3$, number densities $n_g \sim 10^{-4} - 10^{-3}\,h^3{\rm Mpc}^{-3}$, redshift range $0.5 < z < 2.0$, multi-tracer bias splitting.

- **GW sirens:** $N_{\rm siren} = 50-500$ with EM counterparts, $\sigma_t = 1-10\,$s, redshift reach $z \lesssim 2$.

Final constraints scale with these knobs in near-factorizable ways through $\mathcal{S}_i$.

Deliverables. We will provide (i) corner plots for $\{n_s, r, f_{\rm NL}, g_*, c_T\}$ under individual and joint probes, (ii) sensitivity maps $\mathcal{S}_i$, (iii) degeneracy diagnostics and principal components, and (iv) emulator/SBI validation of Fisher errors in the non-Gaussian/anisotropic regime.

Summary. Fisher and simulation-based pipelines together enable decision-grade forecasts for $(n_s, r, f_{\rm NL}, g_*, c_T)$. Large-scale LSS (for $f_{\rm NL}, g_*$), delensed CMB B-modes (for r), and multi-messenger timing (for c_T) form a complementary triad; their combination breaks key degeneracies and charts a clear discovery space for spectral cosmology.

15.2 Dark-Sector Probes: $w_{\rm eff}(t)$ and Environment Couplings

Objective and parameter set

Residual, ultra–long relaxation of the spectral–viscosity field η induces a late–time, mildly time–dependent effective equation of state,

$$w_{\rm eff}(t) \;=\; -1+\delta(t), \qquad |\delta(t)| \ll 1, \tag{15.9}$$

and, generically, an *environmental response* of the dark–energy (DE) density to the matter density field δ_m, inherited from spectral couplings discussed in Vol. I and Chaps. 8–9. We forecast constraints on

$$\boldsymbol{\vartheta} \;=\; \left\{\, w_0,\; w_a\,;\; A_{\rm env},\; \nu_{\rm env},\; k_c \,\right\},$$

with (w_0, w_a) the CPL background parameters $w(a) = w_0 + w_a(1-a)$ (a convenient proxy for $\delta(t)$), and $(A_{\rm env}, \nu_{\rm env}, k_c)$ an *environment kernel* describing DE–matter correlations (defined below). All forecasts marginalize the primordial set $\{n_s, r, f_{\rm NL}, g_*\}$ where relevant.

Spectral mapping and phenomenological ansatz

In the spectral aether, the background shift $\delta(t)$ scales as

$$\delta(t) \;\propto\; \frac{\dot\eta^2}{V_\eta(\eta)+\cdots} \;+\; \mathcal{O}\big(\nabla\eta\cdot\nabla\eta\big), \tag{15.10}$$

yielding $w_{\rm eff}(t) \gtrsim -1$ with mild time evolution. Spatially, residual couplings induce a linear–response relation between DE fluctuations and matter,

$$\frac{\delta\rho_{\rm DE}}{\bar\rho_{\rm DE}}(\mathbf{k},z) \;=\; K_{\rm env}(k,z)\,\delta_m(\mathbf{k},z), \qquad K_{\rm env}(k,z) \;=\; A_{\rm env}\left(\frac{k}{k_c}\right)^{\nu_{\rm env}} \exp\!\left[-\left(\frac{k}{k_c}\right)^2\right], \tag{15.11}$$

where $A_{\rm env}$ is an amplitude, k_c a correlation scale set by spectral diffusion, and $\nu_{\rm env}$ a slope (e.g. dipole/quadrupole–like response at the largest scales). Eq. (15.11) captures the leading DE–matter cross–power in the linear regime,

$$P_{m\,\rm DE}(k,z) \;=\; K_{\rm env}(k,z)\,P_{mm}(k,z), \qquad P_{\rm DEDE}(k,z) \;=\; K_{\rm env}(k,z)^2\,P_{mm}(k,z). \tag{15.12}$$

This ansatz is agnostic about microphysics but consistent with the residual η relaxation picture (Sec. 8).

Background probes: distances and expansion

Late–time distances constrain (w_0, w_a) through

$$H^2(z) = H_0^2\left[\Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_k(1+z)^2 + \Omega_{\rm DE}\,\exp\!\Big(3\int_0^z dz'\,\frac{1+w(z')}{1+z'}\Big)\right], \tag{15.13}$$

with $w(z) = w_0 + w_a z/(1+z)$ in CPL form. We use compressed likelihoods for BAO ($D_M(z)/r_d$, $H(z)r_d$), SNe ($\mu(z)$), and standard sirens ($D_L(z)$), assembling a Gaussian Fisher block

$$F_{ij}^{\rm bg} = \sum_{\alpha,\beta} \frac{\partial \mathbf{d}_\alpha}{\partial\vartheta_i}\,\big[\mathbf{C}^{-1}\big]_{\alpha\beta}\,\frac{\partial \mathbf{d}_\beta}{\partial\vartheta_j}, \qquad \mathbf{d}\in\{D_M/r_d,\; Hr_d,\; \mu,\; D_L\}. \tag{15.14}$$

We include curvature and H_0 as nuisances with broad priors, or impose $\Omega_k \simeq 0$ per *Planck*/BAO if desired.

Growth and lensing with environment couplings

On linear scales, the matter growth factor $D(a)$ obeys

$$D'' + \left[\frac{3}{a} + \frac{d\ln H}{d\ln a}\right] D' - \frac{3}{2}\frac{\Omega_m(a)}{a^2}\,\mu(a,k)\,D = 0, \tag{15.15}$$

where $\mu(a,k)$ captures any effective modification of clustering sourced by DE inhomogeneities. In our framework,

$$\mu(a,k) = 1 + \frac{\Omega_{\rm DE}(a)}{\Omega_m(a)}\, K_{\rm env}(k,a)\, \Xi(a), \tag{15.16}$$

with $\Xi(a)$ an order–unity time–window function encoding the efficiency of DE response to δ_m (vanishing at high z, saturating at late times). Weak lensing probes the Weyl potential, leading to a convergence power spectrum between source bins i,j,

$$C_\ell^{\kappa\kappa,ij} = \int_0^{\chi_H} \frac{d\chi}{\chi^2}\, W_i(\chi) W_j(\chi)\, P_{\Phi+\Psi}\!\left(k = \frac{\ell+1/2}{\chi}, z(\chi)\right), \tag{15.17}$$

with

$$P_{\Phi+\Psi}(k,z) \;=\; \frac{9}{4}\,\frac{H_0^4\,\Omega_m^2}{(1+z)^2}\,\frac{P_{mm}(k,z)}{k^4}\,\Sigma(a,k), \qquad \Sigma(a,k) = 1 + \frac{\Omega_{\rm DE}}{\Omega_m}\, K_{\rm env}(k,a)\,\Upsilon(a). \tag{15.18}$$

Here Σ parallels μ for lensing; $\Upsilon(a)$ is a second window (equal to Ξ in the simplest cases). Redshift–space distortions (RSD) measure $f\sigma_8(a)$ with $f \equiv d\ln D/d\ln a$, sensitive to $\mu(a,k)$ via Eq. (15.15).

ISW–LSS and multi–probe cross–correlations

The late–time Integrated Sachs–Wolfe (ISW) effect correlates CMB temperature with LSS tracers:

$$C_\ell^{Tg} = \int_0^{\chi_*} \frac{d\chi}{\chi^2}\, W_{\rm ISW}(\chi)\, W_g(\chi)\, P_{\Phi' g}\!\left(k = \frac{\ell+1/2}{\chi}, z(\chi)\right), \qquad W_{\rm ISW} \propto \frac{d}{d\eta}\big(\Phi+\Psi\big), \tag{15.19}$$

where primes denote conformal–time derivatives. Since $K_{\rm env} \neq 0$ modifies the decay of $\Phi+\Psi$ at late times (via Σ), C_ℓ^{Tg} is a sensitive, *low–systematics* handle on $(A_{\rm env}, k_c)$.

Tomographic Fisher for WL+GC+RSD with DE response

For tomographic galaxy (GC) and shear (WL) fields with kernels $W_X^i(\chi)$, we assemble

$$C_\ell^{XY,ij} = \int_0^{\chi_H} \frac{d\chi}{\chi^2}\, W_X^i(\chi) W_Y^j(\chi)\, P_{XY}\!\left(k = \frac{\ell+1/2}{\chi}, z(\chi)\right), \quad X,Y \in \{g,\kappa\}, \tag{15.20}$$

with

$$P_{gg} = b_i b_j P_{mm}, \quad P_{g\kappa} \propto b_i\, \Sigma\, P_{mm}, \quad P_{\kappa\kappa} \propto \Sigma\, P_{mm}.$$

RSD enters via anisotropic GC spectra or compressed $f\sigma_8(z)$ measurements. The multi–probe Fisher is

$$F_{ij}^{\rm tomo} = \sum_\ell \frac{(2\ell+1)\, f_{\rm sky}}{2}\, {\rm Tr}\big[\mathbf{C}_\ell^{-1}\, \partial_i \mathbf{C}_\ell\, \mathbf{C}_\ell^{-1}\, \partial_j \mathbf{C}_\ell\big], \tag{15.21}$$

where $\mathbf{C}_\ell$ stacks all (g,κ) auto– and cross–spectra across redshift bins with shot–noise and shape–noise contributions added on the diagonal. Derivatives $\partial_i \mathbf{C}_\ell$ propagate sensitivities to (w_0, w_a) through $H(z), D_A(z)$ and to $(A_{\rm env}, \nu_{\rm env}, k_c)$ through μ, Σ in Eqs. (15.16)–(15.18). Galaxy bias, photo–z, intrinsic alignments, and shear calibration are included as nuisance parameters with priors.

Degeneracies and strategies to break them

Key degeneracies include:

- (w_0, w_a) *vs.* Ω_k: broken by combining SNe/BAO with standard sirens and high–z BAO.
- $(A_{\rm env}, k_c)$ *vs.* σ_8: broken by WL+GC cross–spectra and ISW–LSS, which respond differently to Σ and σ_8.
- $(A_{\rm env}, \nu_{\rm env})$ *vs.* M_ν: massive neutrinos suppress small–scale power; scale–dependence of $K_{\rm env}$ at $k \lesssim k_c$ plus very–large–scale clustering (multi–tracer) separates the effects.

Additional handles: cluster counts with self–calibration, the E_G statistic (combining WL and RSD) sensitive to Σ/μ, and tomographic ISW with spectroscopic LSS.

Sensitivity heuristics and survey leverage

Define per–mode information densities

$$\mathcal{S}^{\rm bg}_{w_0,w_a}(z) \sim \left|\frac{\partial \ln D_M}{\partial(w_0,w_a)}\right| \sigma(D_M)^{-1} + \left|\frac{\partial \ln H}{\partial(w_0,w_a)}\right| \sigma(H)^{-1}, \tag{15.22}$$

and

$$\mathcal{S}^{\rm LSS}_{\rm env}(k,z) \sim \left.\frac{|\partial_{A_{\rm env}} \ln C_\ell^{\kappa\kappa}| + |\partial_{A_{\rm env}} \ln C_\ell^{g\kappa}|}{\sigma[C_\ell]}\right|_{k=(\ell+1/2)/\chi(z)}. \tag{15.23}$$

Late–time, wide–area imaging (for WL) and dense spectroscopy (for RSD, ISW–LSS) maximize $\mathcal{S}_{\rm env}$ on large angular scales; high–accuracy BAO and sirens maximize $\mathcal{S}_{w_0,w_a}$.

Systematics and priors as gates

We impose hard/soft priors motivated by Volume I gates:

$$|w_{\rm eff}(t)+1| = |\delta(t)| \le \delta_{\rm max} \ll 1, \qquad |A_{\rm env}| \le A_{\rm max}, \quad 0 \le k_c \le k^{\rm lin}_{\rm max}, \tag{15.24}$$

and subluminal/causality constraints inherited from the curvature sector. Shear calibration, photo–z biases, and intrinsic alignments receive Gaussian priors consistent with survey design; *all* nuisance parameters are sampled jointly to produce robust, gate–aware constraints.

Deliverables. We will present: (i) joint posteriors in (w_0, w_a) and $(A_{\rm env}, \nu_{\rm env}, k_c)$ under WL+GC+RSD+ISW+BAO+SI (ii) PCA modes of $w(z)$ and $K_{\rm env}(k)$ with resolution kernels, (iii) E_G and ISW–LSS diagnostics isolating environment couplings, and (iv) decision plots showing where spectral cosmology is distinguishable from ΛCDM (i.e. $A_{\rm env} = 0$, $w_a = 0$) at given survey specifications.

Summary. Late–time acceleration from residual η–relaxation is testable via *background* distances (w_0, w_a) and *environment* couplings encoded in μ, Σ. Tomographic WL+GC+RSD, augmented by ISW–LSS and standard sirens, provides complementary sensitivity to both pillars. The kernel $K_{\rm env}(k)$ offers a distinctive, falsifiable signature of spectral cosmology beyond a rigid cosmological constant.

15.3 Defect and Resonance Signatures in LSS/Lensing

Signal model from spectral defect networks

Phase transitions seeded by the η–wave (Chs. 5, 7) generate a mixed population of solitonic defects: domain walls (π_0), strings/vortices (π_1), monopoles (π_2), and Hopf/Skyrme textures (π_3). Coarse–graining over a comoving correlation length $\xi(z)$ and core scale $\ell \ll \xi$, we parameterize the defect energy density fraction $f_{\rm def} \equiv \bar\rho_{\rm def}/\bar\rho_m$ and a *shape kernel* $\mathcal{W}_{\rm def}(k)$ encoding core structure and tension $\mathcal{T}$:

$$P_{mm}(k,z) \;=\; P^{\Lambda\rm CDM}_{mm}(k,z) \;+\; A_{\rm def}(z)\,\mathcal{W}_{\rm def}(k\xi)\,, \qquad A_{\rm def}(z) \;\propto\; f_{\rm def}(z)\,\frac{\mathcal{T}}{\bar\rho_m(z)}\,. \tag{15.25}$$

For strings, $\mathcal{W}_{\rm def}(q) \sim (1+q^2)^{-1}$ (white–noise to $k \sim \xi^{-1}$ then k^{-2}); for walls, $\mathcal{W}_{\rm def}(q) \sim (1+q^2)^{-3/2}$; for textures, a broad bump at $k \sim \xi^{-1}$ with k^{-4} tail. Nonlinear mode–coupling with the resonance field $\mathcal{I}(\mathbf{x})$ (Ch. 7) induces a bispectrum

$$B^{\rm def}_{mmm}(k_1,k_2,k_3) \;=\; A_B(z)\,\Big[\mathcal{W}_{\rm def}(k_1\xi)\mathcal{W}_{\rm res}(k_2\xi) \;+\; 2 \text{ perms}\Big], \tag{15.26}$$

with $\mathcal{W}_{\rm res}(q)$ peaked near the most–unstable band of Eq. (V2.**??**). Shape diagnostics discriminate from standard local/equil./orthogonal templates.

Weak lensing and defect–induced B–modes

Defect stress anisotropy sources a curl component of the lensing deflection and a *lensing B–mode* at leading order. Writing the convergence and rotation power spectra

$$\{C_\ell^{\kappa\kappa},\, C_\ell^{\omega\omega}\} \;=\; \int_0^{\chi_H} \frac{d\chi}{\chi^2}\, W_\kappa^2(\chi)\, \left\{ \Sigma^2(k,z)\, \frac{9H_0^4\Omega_m^2}{4(1+z)^2}\, \frac{P_{mm}(k,z)}{k^4},\quad \mathcal{R}^2(k,z)\, \frac{9H_0^4\Omega_m^2}{4(1+z)^2}\, \frac{P_{mm}(k,z)}{k^4} \right\}_{k=(\ell+1/2)/\chi}, \tag{15.27}$$

defects contribute through $\Sigma = 1 + \cdots$ (as in Sec. 15.2) and a new rotation kernel $\mathcal{R}(k,z)$ which vanishes in ΛCDM to linear order. For a straight–string toy model with line density $n_{\rm str}$ and tension parameter $\mathcal{T}$,

$$\mathcal{R}^2(k,z) \;\simeq\; \frac{\pi}{2}\, n_{\rm str}(z)\, \left(\frac{\mathcal{T}}{\bar\rho_m}\right)^2 \frac{(k\ell)^2}{\left(1+k^2\ell^2\right)^2}, \tag{15.28}$$

peaked at $k \sim \ell^{-1}$. A detection of $C_\ell^{\omega\omega} > 0$ consistent across source bins would be a smoking–gun signature of defect networks.

Morphological statistics: Minkowski functionals, genus, skeleton

Beyond two–point functions, *morphology* isolates filamentary and wall–like imprints:

- **3D Minkowski functionals** of a smoothed density field $\delta(\mathbf{x})$ at threshold $\nu \equiv \delta/\sigma_0$: volume V_0, surface area V_1, integrated mean curvature V_2, and Euler characteristic (genus) V_3. For a Gaussian random field (GRF) with spectral moments $\sigma_j^2 = \int \frac{d^3k}{(2\pi)^3}\, k^{2j} P(k) W^2(kR)$,

$$V_0^{\rm GRF}(\nu) = \frac{1}{2}\,{\rm erfc}\!\left(\frac{\nu}{\sqrt{2}}\right), \quad V_1^{\rm GRF}(\nu) = \frac{1}{6\pi^2}\left(\frac{\sigma_1}{\sqrt{3}\sigma_0}\right) e^{-\nu^2/2},$$
$$V_2^{\rm GRF}(\nu) = \frac{1}{6\pi^2}\left(\frac{\sigma_1}{\sqrt{3}\sigma_0}\right)^2 \nu\, e^{-\nu^2/2}, \quad V_3^{\rm GRF}(\nu) = \frac{1}{(2\pi)^2}\left(\frac{\sigma_1}{\sqrt{3}\sigma_0}\right)^3 (\nu^2-1)\, e^{-\nu^2/2}. \tag{15.29}$$

  Defects/resonance shift (σ_0, σ_1) and add non–Gaussian corrections $\Delta V_\mu(\nu)$ whose *shape* (e.g. excess walls $\Rightarrow$ skewed V_2, enhanced filamentarity $\Rightarrow$ specific ν–dependence of V_1) discriminates scenarios.

- **Genus and Betti numbers** in redshift slices provide topological counts of holes/components; domain–wall–like signals elevate the number of closed surfaces at high thresholds (altering the genus curve zero–crossings).

- **Skeleton length density** $\mathcal{L}_{\rm skel}(R)$ (total length of ridges per volume, smoothed at scale R) scales with a distinct combination of spectral moments; analytic GRF predictions exist, and defect/resonance add an excess at $R \sim \xi$.

Matched filters and filament/string finders

We implement harmonic–space matched filters for thin structures:

$$\hat{A}_{\rm fil} = \frac{\sum_{\mathbf{k}} \tilde{T}^*(\mathbf{k})\, \tilde{\delta}(\mathbf{k})/P_{\rm tot}(k)}{\sum_{\mathbf{k}} |\tilde{T}(\mathbf{k})|^2/P_{\rm tot}(k)}, \qquad \tilde{T}_{\rm wall}(k) \propto \frac{1}{1+k^2\ell^2}, \quad \tilde{T}_{\rm string}(k_\perp) \propto \frac{1}{1+k_\perp^2\ell^2}, \tag{15.30}$$

with $P_{\rm tot} = P_{mm} + P_{\rm noise}$. The estimator yields an *amplitude map* and S/N for walls/strings at candidate scales ℓ; cross–correlating $\hat{A}_{\rm fil}$ with lensing maps (and with the resonance proxy $\hat{\mathcal{I}}$ from Ch. 7) further suppresses systematics.

Line–intensity mapping (LIM) prospects

Spectral resonance predicts large–scale modulations in the neutral gas/star–formation tracers. For an LIM field I_X (e.g. HI 21 cm, CO, [CII]) with bias $b_X(z)$,

$$P_{I_X I_X}(k,z) = \bar{T}_X^2\, b_X^2\, P_{mm}(k,z) + P_{\rm shot}^X, \qquad P_{I_X g}(k,z) = \bar{T}_X\, b_X\, b_g\, P_{mm}(k,z), \tag{15.31}$$

so the additive term in Eq. (15.25) appears directly in auto– and cross–spectra. LIM × lensing tomography (CMB–κ, galaxy–κ) isolates the large–scale wall/filament power at $k \sim \xi^{-1}$ with minimal stellar–mass selection biases. Forecast S/N for an amplitude parameter $A_{\rm def}$ reads

$$\left(\frac{S}{N}\right)^2 = \sum_{\ell,z} \frac{\left[\partial_{A_{\rm def}} C_\ell^{XY}(z)\right]^2}{\mathrm{Var}[C_\ell^{XY}(z)]}, \quad X, Y \in \{I_X,\ g,\ \kappa\}, \tag{15.32}$$

with multi–tracer $X \neq Y$ combinations particularly efficient on the largest angular scales.

Forecast machinery and degeneracy control

We augment the LSS+WL Fisher (Sec. 15.2) by $\boldsymbol{\phi}_{\rm def} = \{A_{\rm def}, \xi, \ell, \mathrm{type}\}$ with discrete *type* labeling (wall/string/texture) handled via Bayesian evidence or information criteria. Principal–component decompositions of $\Delta V_\mu(\nu)$, $\mathcal{L}_{\rm skel}(R)$, and LIM cross–spectra define near–orthogonal summary statistics. Baryonic feedback degeneracies are mitigated by:

1. focusing on $k \lesssim 0.3\, h\,\mathrm{Mpc}^{-1}$ for two–point functions and using morphology (less sensitive to small–scale feedback),

2. cross–correlating with lensing (gravity–only) and with LIM (gas–weighted) to separate true geometry from astrophysics,

3. imposing spectral gates from Vol. I (causality, subluminality, stability) as priors on $(\xi, \ell, \mathcal{T})$.

Deliverables and discovery criteria

We will provide (i) joint posteriors for $\{A_{\rm def}, \xi, \ell\}$ across probes; (ii) model selection odds between {wall,string,texture} and "no–defect"; (iii) morphology residuals $\Delta V_\mu(\nu)$ and skeleton excess $\Delta\mathcal{L}_{\rm skel}(R)$ with covariance; (iv) LIM×LSS×WL S/N maps for ξ–scale features; and (v) a *decision table* flagging discovery if either (a) $C_\ell^{\omega\omega}$ is detected at $\geq 5\sigma$ with the scale–dependence (15.28) *and* consistent ξ from morphology, or (b) a coherent excess at $k \sim \xi^{-1}$ appears in multi–probe cross–spectra with Bayes factor $\ln \mathcal{B} > 5$ over ΛCDM+baryons.

Summary. Defect and resonance physics imprint *distinctive, multi–scale* signals: additive broad–band power near $k \sim \xi^{-1}$, a lensing rotation/B–mode, and robust morphological anomalies. A coordinated analysis of WL/GC/LIM with topology–aware statistics yields a falsifiable path to confirm or rule out the spectral–cosmology origin of structure.

15.4 Decision Tree for Near/Mid-Term Tests

Objective and scope

We formalize a *probe–driven* decision tree that maps forecasted sensitivities (Sec. 15.1) and defect/resonance discriminants (Sec. 15.3) into concrete *pass/fail* outcomes for the spectral–cosmology hypotheses:

$$\mathcal{H}_{\rm SC} \equiv \{\eta\text{–wave inflation, spectral anisotropy, defects/resonances, residual } \eta\}\,, \quad \mathcal{H}_{\Lambda\rm CDM} \equiv \{\text{single-clock, } c_T = 1\text{, no de}$$

We adopt *pre-registered* criteria based on signal-to-noise (S/N), Bayes factors $\ln \mathcal{B}$, and frequentist p–values, with consistency gates (Vol. I) enforced as priors.

Inputs (data epochs and thresholds)

Let the near-term horizon be $T_{\rm near} \sim 3\,\mathrm{yr}$ (DESI, ACT/SPT, HSC/LSST–Y3, Adv. LIGO/Virgo/KAGRA, PTAs), and the mid-term $T_{\rm mid} \sim 5$–$8\,\mathrm{yr}$ (Euclid, Roman, Rubin full depth, CMB–S4,

SKA/HD–PTA). We collect:

$$\text{Primordial: } \{\hat{n}_s,\ \hat{r},\ \widehat{f_{\rm NL}^{\rm shape}},\ \hat{g}_*,\ \widehat{\delta c_T}\}, \tag{15.33}$$

$$\text{LSS/Lensing: } \{\widehat{C}_\ell^{\kappa\kappa},\ \widehat{C}_\ell^{\omega\omega},\ \widehat{P}_{gg},\ \widehat{P}_{g\kappa},\ \widehat{P}_{\rm LIM}\}, \tag{15.34}$$

$$\text{Morphology: } \{\widehat{V}_\mu(\nu),\ \widehat{\mathcal{L}}_{\rm skel}(R),\ \widehat{\rm genus}(\nu)\}, \tag{15.35}$$

$$\text{GW propagation: } \{\widehat{c}_T(z),\ \widehat{\nu}_{\rm GW}(z)\}, \tag{15.36}$$

$$\text{Dark sector: } \{\widehat{w}_{\rm eff}(z),\ \widehat{\Sigma}(k,z)\}. \tag{15.37}$$

Thresholds adopt conventional discovery standards:

$$\text{detected if S/N} \geq 5 \quad \text{or} \quad \ln\mathcal{B} \geq 5 \quad \text{or} \quad p \leq 3\times 10^{-7}\ (5\sigma),$$

and "strong evidence" if $\ln\mathcal{B} \geq 8$.

Branch A: Primordial scalar/tensor tests

A1: Tensor amplitude and speed.

1. If $\hat{r} \geq r_{\rm th}$ with $r_{\rm th} = 0.01$ (CMB–S4 target) and the pivot tilt obeys $\hat{n}_s - 1 = -(2\hat{\epsilon}_\eta + \hat{\eta}_\eta + \hat{s}_c) \pm \sigma$, mark `PASS-A1`. Else `DEFER-A1`.

2. If $|\widehat{\delta c_T}| \leq 10^{-15}$ (multimessenger, low z) and forecasted early–time $|\delta c_T| < 10^{-3}$, mark `CONSIS-A1`. If $|\widehat{\delta c_T}| > 10^{-15}$ at cosmological z but consistent with spectral curvature bounds, mark `TENT-A1`; otherwise `FAIL-A1`.

A2: Anisotropy and NG shape.

1. If $|\hat{g}_*| \geq 0.01$ with angle pattern predicted by $c_s(\theta)$ (Sec. 4.3), `PASS-A2`; if null with forecast sensitivity $\sigma(g_*) \leq 0.005$, `CONSIS-A2`.

2. If a non–Gaussian bispectrum is detected with $\ln\mathcal{B}[f_{\rm NL}^{\rm eff} > 0\,(\text{anisotropic}) : \text{standard templates}] \geq 5$, `PASS-A2`. Else `DEFER-A2`.

Branch B: Late–time and dark sector

B1: Time–dependent equation of state. If a joint fit (Sec. 15.2) yields

$$\delta(z) \equiv 1 + w_{\rm eff}(z) = \delta_0 + \delta_1 \frac{z}{1+z}$$

with $(\delta_1/\sigma_{\delta_1}) \geq 3$ and LIM/lensing cross–correlations favor environment–dependence (*nonzero* $d\delta/d\ln\rho$ at $\geq 3\sigma$), mark `PASS-B1`. If $|\delta_1| < \sigma_{\delta_1}$ but bounds are consistent with residual η predictions, `CONSIS-B1`; otherwise `DEFER-B1`.

B2: Growth and slip. If $\Sigma(k,z) - 1$ departs from zero at $k \lesssim 0.1\,h\,{\rm Mpc}^{-1}$ with the spectral template (Eq. (V2.**??**)) at S/N ≥ 5 *and* remains consistent with subluminality and no–ghost priors, `PASS-B2`; else `DEFER-B2`.

Branch C: Defects and resonance

C1: Lensing rotation and B–modes. If $C_\ell^{\omega\omega}$ is detected with the scale–dependence of Eq. (V2.15.28) at $\geq 5\sigma$, `PASS-C1`; if $C_\ell^{\omega\omega}$ null with $\sigma(C_\ell^{\omega\omega})$ below the minimal spectral prediction, `FAIL-C1`; otherwise `DEFER-C1`.

C2: Morphology and matched filters. If Minkowski/skeleton residuals $\Delta V_\mu(\nu)$ and $\Delta\mathcal{L}_{\rm skel}(R)$ jointly prefer $\xi \sim k^{-1}$ with $\ln\mathcal{B}[\text{defect/resonance} : \ \Lambda\text{CDM+baryons}] \geq 5$, and matched–filter amplitudes $\hat{A}_{\rm fil}$ cross–correlate with lensing at $\geq 4\sigma$, `PASS-C2`; else `DEFER-C2`.

Cross–probe logic and escalation

Define binary flags $\mathbb{I}_{\mathrm{A1}}, \mathbb{I}_{\mathrm{A2}}, \mathbb{I}_{\mathrm{B1}}, \mathbb{I}_{\mathrm{B2}}, \mathbb{I}_{\mathrm{C1}}, \mathbb{I}_{\mathrm{C2}} \in \{0, 1\}$ for PASS. The *global verdict* uses a weighted score

$$\mathcal{S}_{\mathrm{global}} = \sum_{X \in \{\mathrm{A1},\ldots,\mathrm{C2}\}} w_X \, \mathbb{I}_X, \qquad \sum_X w_X = 1, \tag{15.38}$$

with $(w_{\mathrm{A1}}, w_{\mathrm{A2}}, w_{\mathrm{B1}}, w_{\mathrm{B2}}, w_{\mathrm{C1}}, w_{\mathrm{C2}}) = (0.22, 0.18, 0.18, 0.12, 0.18, 0.12)$ (primordial amplitude and late–time/GW/lensing carry slightly higher weight). We declare:

`DISCOVERED` if $\mathcal{S}_{\mathrm{global}} \geq 0.6$, `SUPPORTED` if $0.4 \leq \mathcal{S}_{\mathrm{global}} < 0.6$, `INCONCLUSIVE` if $0.25 \leq \mathcal{S}_{\mathrm{global}} < 0.4$, `DISFAVORED` i

Escalation rules:

- If `PASS-A1` and `PASS-A2` but `DEFER` in B/C: prioritize CMB–S4 polarization delensing and primordial–NG shape reconstruction; schedule LIM surveys to break B–branch degeneracies.
- If `PASS-B1` or `B2` without A–branch signals: prioritize WL/LSS tomography, PTAs for $c_T(z)$, and targeted environment–dependence tests; reduce emphasis on r forecasts.
- If `PASS-C1/C2` alone: expand morphology program (3D genus, Betti, skeleton) and develop end–to–end *defect–forward* mocks; request deep–wide shear and LIM time for confirmation.

Risk matrix and mitigation

Risk	Impact	Likelihood	Mitigation
Systematic B–modes in shear	High	Medium	multi–band shear, null tests, star–galaxy PSF splits
Anisotropic survey window	Medium	High	window deconvolution, mock–based covariance
Baryonic feedback degeneracy	Medium	High	morphology stats, WL×LIM cross–checks
PTA clock/ephemeris errors	High	Medium	JPL ephemeris marginalization, multi–array joint fits
Foregrounds in CMB/IM	High	Medium	component separation with parametric + GNILC hybrids

Reporting and pre-registration

For each branch, we pre-register:

1. the exact summary statistics and estimators,
2. the *a priori* thresholds $(\mathrm{S/N}, \ln \mathcal{B}, p)$,
3. the nuisance model and priors (including gates S/G/TQ/D),
4. the blinding scheme and unblinding triggers.

Final claims will quote both Bayesian evidences and frequentist significances, provide open covariances, and release analysis code and synthetic validation sets (Ch. 11).

Summary. This decision tree turns the forecast landscape into a *test plan*: clear pass/fail nodes for primordial, late–time, and defect/resonance predictions, weighted into a global verdict with escalation paths. It ensures that spectral cosmology is confronted with data in a falsifiable, transparent, and reproducible manner over the near- and mid-term horizons.

Part VIII

Technical Appendices for Spectral Cosmology

Appendix A: Quadratic Action and Mode Decomposition with Anisotropy

Gauge Choices, Constraints, and Canonical Variables

We work with the *emergent* metric $g^{\rm eff}_{\mu\nu}(S)$ (Vol. I) and decompose w.r.t. the aether frame u^μ:

$$g^{\rm eff}_{\mu\nu}\,dx^\mu dx^\nu = -N^2 dt^2 + h_{ij}\big(dx^i + N^i dt\big)\big(dx^j + N^j dt\big), \qquad h_{ij} = a^2(t)\,(\delta_{ij} + \gamma_{ij}),$$

with γ_{ij} transverse-traceless for tensors. An internal *director* $\hat\chi^i$ (unit vector) encodes mild statistical anisotropy. We adopt the **comoving–aether gauge**

$$\delta T = 0, \qquad \delta q \equiv -T^0{}_i \partial^{-2}\partial^i = 0,$$

so that the single scalar d.o.f. at long wavelengths is the curvature perturbation ζ defined by

$$h_{ij} = a^2(t)\,e^{2\zeta}\,(\delta_{ij} + \gamma_{ij})\,, \qquad \partial^i\gamma_{ij} = 0, \quad \gamma^i{}_i = 0.$$

Lapse and shift are nondynamical at quadratic order; expanding the unified action of Vol. I around the inflating background (Vol. II, Sec. **??**), their constraint equations yield

$$\frac{\delta S}{\delta N} = 0 \;\Rightarrow\; N = 1 + \alpha_1\,\dot\zeta + \alpha_2\,\frac{\partial_\parallel^2\zeta}{a^2 H} + \alpha_3\,\frac{\partial_\perp^2\zeta}{a^2 H} + \cdots\,, \tag{15.39}$$

$$\frac{\delta S}{\delta N^i} = 0 \;\Rightarrow\; N_i = \partial_i\psi, \quad \psi = \beta_1\,\zeta + \beta_2\,\frac{\dot\zeta}{H} + \beta_{3,\parallel}\,\frac{\partial_\parallel\zeta}{aH} + \beta_{3,\perp}\,\frac{\partial_\perp\zeta}{aH} + \cdots\,, \tag{15.40}$$

where $\partial_\parallel \equiv \hat\chi^i\partial_i$ and $\partial_\perp^2 \equiv \partial^2 - \partial_\parallel^2$. The coefficients α, β are functions of spectral slow parameters $(\epsilon_\eta, \eta_\eta)$ and EFT couplings (Vol. II, Sec. **??**). Eliminating (N, N^i) defines the *canonical variables*

$$v_\zeta \equiv z(\theta)\,\zeta, \qquad z^2(\theta) = 2a^2 M_{\rm eff}^2(\rho)\,\frac{\epsilon_\eta}{c_s^2(\theta)}; \qquad v_T^{(s)} \equiv \frac{aM_{\rm eff}}{2}\,h^{(s)},\; s = +, \times,$$

with the *anisotropic* sound speed

$$c_s^2(\theta) = c_\perp^2 + \big(c_\parallel^2 - c_\perp^2\big)\cos^2\theta, \qquad \cos\theta \equiv \frac{\mathbf{k}\cdot\hat\chi}{k}.$$

Derivation of $\mathcal{L}^{(2)}$ and Principal Symbols

To quadratic order, the scalar and tensor actions reduce to

$$S_\zeta^{(2)} = \frac{1}{2}\int d^4x\,a^3\,\mathcal{Q}_\zeta(\theta)\left[\dot\zeta^2 - c_s^2(\theta)\,\frac{(\partial_i\zeta)^2}{a^2}\right], \quad \mathcal{Q}_\zeta(\theta) = \frac{2M_{\rm eff}^2(\rho)\,\epsilon_\eta}{c_s^2(\theta)}, \tag{15.41}$$

$$S_T^{(2)} = \frac{M_{\rm eff}^2}{8}\int d^4x\,a^3\left[\dot h_{ij}^2 - c_T^2\,\frac{(\partial_k h_{ij})^2}{a^2}\right], \qquad c_T^2 = 1 + \delta c_T(\mathcal{S}). \tag{15.42}$$

In Fourier space, the principal symbol of the scalar sector is

$$\mathsf{P}_\zeta(\omega, \mathbf{k}; \theta) = \mathcal{Q}_\zeta(\theta)\,\big(-\omega^2 + c_s^2(\theta)\,k^2\big),$$

while for tensors

$$\mathsf{P}_T(\omega, \mathbf{k}) = \frac{M_{\rm eff}^2}{4}\,\big(-\omega^2 + c_T^2 k^2\big)\,\mathbb{K}_{\rm TT}.$$

The canonical mode functions $v_k(\tau;\theta)$ satisfy

$$v_k'' + \left(c_s^2(\theta)k^2 - \frac{z''}{z}\right) v_k = 0, \qquad {}' \equiv \frac{d}{d\tau}, \quad d\tau = \frac{dt}{a},$$

with $z = a\sqrt{2\epsilon_\eta}M_{\rm eff}/c_s(\theta)$ and Bunch–Davies initial conditions $v_k \to e^{-ic_s k\tau}/\sqrt{2c_s k}$ at $-c_s k\tau \gg 1$. The power spectrum at freeze-out $c_s k = aH$ is

$$\mathcal{P}_\zeta(k,\theta) = \frac{H^2}{8\pi^2 M_{\rm eff}^2\,\epsilon_\eta\,c_s(\theta)}\Big|_{c_s k = aH}.$$

Positivity, Hyperbolicity, and Sub-Luminality Gates

The quadratic theory passes the global S/G/TQ/D gates iff:

1. **No-ghost** (positivity): $\mathcal{Q}_\zeta(\theta) > 0$ and $M_{\rm eff}^2 > 0$ for all angles θ.
2. **Hyperbolicity/causality**: principal cones lie inside the $g^{\rm eff}$ light cone,

   $$0 < c_T^2 \le 1, \qquad 0 < c_\perp^2 \le c_\parallel^2 \le 1,$$

   and the anisotropy $|c_\parallel^2 - c_\perp^2| \ll 1$ during inflation/CMB generation.
3. **Gradient stability**: $c_s^2(\theta) > 0$ for all θ, and $z''/z > 0$ on superhorizon scales (slow-roll).
4. **Restoration**: in the post-inflationary limit, $c_T \to 1$ and $c_\parallel - c_\perp \to 0$ (Sec. 4.4).

Violations of any item truncate the admissible parameter region in the inference pipelines (Vol. II, Ch. 10).

Appendix B: Cubic Action and Bispectrum Kernels

Interaction Hamiltonian and In-In Formalism

Promoting the unitary-gauge EFT (Vol. II, App. VIII) to cubic order and restoring time diffeos via Stückelberg π yields, after integrating out nondynamical fields,

$$S_\zeta^{(3)} = \int d^4x\, a^3 \Big[A_1(\theta)\, \zeta\, \dot\zeta^2 + A_2(\theta)\, \zeta\, \frac{(\partial_i \zeta)^2}{a^2} + A_3(\theta)\, \dot\zeta\, \frac{\partial_i \zeta\, \partial_i \partial^{-2} \dot\zeta}{a^2} + A_4(\theta)\, \frac{(\partial_\parallel \zeta)^2\, \zeta}{a^2} + A_5(\theta)\, \frac{(\partial_\perp \zeta)^2\, \zeta}{a^2} + \cdots \Big], \tag{15.43}$$

where angle dependence arises from $c_s(\theta)$ and director couplings. Possible scalar–director mixing contributes

$$S_{\zeta\delta\chi}^{(3)} = \int d^4x\, a^3 \left[B_1\, \delta\chi_\parallel\, \dot\zeta^2 + B_2\, \delta\chi_\parallel\, \frac{(\partial \zeta)^2}{a^2} + B_3\, \zeta\, \frac{(\partial_\parallel \delta\chi)^2}{a^2} + \cdots \right].$$

The bispectrum in the in–in (Schwinger–Keldysh) formalism is

$$\big\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \big\rangle = -2\ \mathrm{Re}\ \mathrm{i} \int_{-\infty(1-\mathrm{i}\epsilon)}^{0} d\tau\, \big\langle 0 \big| \zeta_{\mathbf{k}_1}(0) \zeta_{\mathbf{k}_2}(0) \zeta_{\mathbf{k}_3}(0)\, H_{\mathrm{int}}(\tau) \big| 0 \big\rangle,$$

with $H_{\mathrm{int}} = -\mathcal{L}^{(3)}$ and Bunch–Davies initial conditions for $\upsilon_k(\tau;\theta)$.

Shape Functions with $c_s(\theta)$ and $\delta\chi$ Mixing

At leading order in slow-roll and small anisotropy, the reduced bispectrum takes the form

$$B_\zeta(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = (2\pi)^4\, \mathcal{P}_\zeta^2\, \frac{\mathcal{S}(k_1, k_2, k_3; \{\hat{\mathbf{k}}_i \cdot \hat\chi\})}{(k_1 k_2 k_3)^2},$$

with a multipole expansion

$$\mathcal{S} = \mathcal{S}_0(k_i) + g_* \sum_n \mathcal{S}_2^{(n)}(k_i) \sum_{a=1}^{3} P_2\big(\hat{\mathbf{k}}_a \cdot \hat\chi\big) + \mathcal{O}(g_*^2),$$

where $g_* \sim (c_\parallel^2 - c_\perp^2)/\bar c_s^2$. For the operators in (15.43) one finds (schematically)

$$\mathcal{S}_0^{\mathrm{equil}} \propto \frac{1}{\bar c_s^2} \left(\sum_i \frac{k_i^3}{K^3} - \sum_{i\neq j} \frac{k_i^2 k_j}{K^3} \right), \qquad K \equiv k_1 + k_2 + k_3, \tag{15.44}$$

$$\mathcal{S}_0^{\mathrm{orth}} \propto \frac{1}{\bar c_s^2} \left(-3 \sum_i \frac{k_i^3}{K^3} + 3 \sum_{i\neq j} \frac{k_i^2 k_j}{K^3} + 8 \frac{k_1 k_2 k_3}{K^3} \right), \tag{15.45}$$

$$\mathcal{S}_2^{(n)} \propto \frac{1}{\bar c_s^2} \Big[\alpha_n(\{k\})\, P_2(\hat{\mathbf{k}}_1 \cdot \hat\chi) + \mathrm{cyc.} \Big]. \tag{15.46}$$

Mixing with $\delta\chi$ adds angle-selective enhancements in folded configurations where $\mathbf{k}_a \parallel \hat{\chi}$. These angular kernels are the targets of the LSS bispectrum and CMB $\langle TTT \rangle$ anisotropy analyses (Vol. II, Sec. 4.3).

IR/UV Control and Soft-Limit Consistency

Soft theorems. Single-clock nonlinearly realized time diffeos imply, in the squeezed limit $q \to 0$,

$$\lim_{q\to 0} B_\zeta(\mathbf{k}, -\mathbf{k}-\mathbf{q}, \mathbf{q}) = -\big(n_s - 1\big)\, P_\zeta(k)\, P_\zeta(q) + \mathcal{O}(g_*\, P_2(\hat{\mathbf{k}}\cdot\hat{\chi})),$$

i.e. the standard consistency relation survives angle-averaging; anisotropy contributes calculable quadrupoles.

IR safety. Long-wavelength redefinitions of ζ (adiabatic modes) are pure gauge at leading order. Using the δN map with anisotropic $c_s(\theta)$ one verifies that infrared secular terms cancel between $\mathcal{L}^{(3)}$ and field redefinitions at tree level.

UV control. The EFT cutoff $\Lambda_{\rm spec}$ obeys

$$H \ll \Lambda_{\rm spec}\,, \qquad \frac{\omega, k}{a} \ll \Lambda_{\rm spec}, \qquad \frac{|c_\parallel^2 - c_\perp^2|}{\bar{c}_s^2} \ll 1.$$

Loop corrections yield counterterms that renormalize $c_s(\theta)$ and the $A_i(\theta)$. A minimal renormalization scheme (dim. regularization in conformal time) preserves the soft limits and keeps $f_{\rm NL}^{\rm eff} \sim \mathcal{O}\big((1-\bar{c}_s^2)/\bar{c}_s^2\big) + \mathcal{O}(g_*)$ under perturbative control.

Parity-odd extensions. If parity-odd/WZ terms (Vol. II, App. VIII) are present, they induce TB/EB correlations and a bispectrum component odd under $\hat{\mathbf{k}} \to -\hat{\mathbf{k}}$ with amplitude suppressed by the corresponding coupling $\vartheta(\bar{\eta})$. These are strongly constrained and serve as clean discovery channels.

Summary. The cubic action inherits a controlled, angle-dependent structure tied to $c_s(\theta)$ and director mixing. Its in–in evaluation produces bispectra with distinctive quadrupolar modulations that (i) reduce to standard single-field results when anisotropy vanishes and (ii) furnish sharp null tests for the spectral cosmology framework.

Appendix C: Anisotropic Kibble–Zurek Scaling

Critical Exponents, Quench Protocols, and Freeze-Out Lengths

Anisotropic critical behavior. Near the instability/critical surface in the spectral medium, we introduce a reduced control parameter $\epsilon(t) \equiv \frac{\lambda(t)-\lambda_c}{\lambda_c}$, where λ is any spectrally controlled knob (e.g. a combination of η, ρ, or a stiffness in the structural sector). Anisotropy with respect to the director $\hat{\chi}$ implies distinct static and dynamic exponents along and transverse to $\hat{\chi}$:

$$\xi_\parallel(\epsilon) = \xi_{0\parallel}\,|\epsilon|^{-\nu_\parallel}, \qquad \xi_\perp(\epsilon) = \xi_{0\perp}\,|\epsilon|^{-\nu_\perp}, \tag{15.47}$$

$$\tau_\parallel(\epsilon) = \tau_{0\parallel}\,|\epsilon|^{-\nu_\parallel z_\parallel}, \qquad \tau_\perp(\epsilon) = \tau_{0\perp}\,|\epsilon|^{-\nu_\perp z_\perp}. \tag{15.48}$$

For relaxational dynamics dominated by spectral diffusion (as in Vol. II, Sec. **??**), one expects $z_{\parallel,\perp} \approx 2$ at leading order, with corrections from coupling to the rolling η condensate.

Quench protocol. We consider a power-law quench across $\epsilon = 0$:

$$\epsilon(t) = \mathrm{sgn}(t)\left|\frac{t}{\tau_Q}\right|^a, \qquad a \geq 1, \tag{15.49}$$

where τ_Q sets the quench rate and $a = 1$ is the standard linear case. The instantaneous rate-of-change timescale is $|\epsilon/\dot{\epsilon}| = \frac{|t|}{a}$.

Freeze-out (KZ) condition—anisotropic form. Modes stop tracking equilibrium when their relaxation time equals the inverse quench rate *in the relevant channel*. We therefore define *direction-resolved* freeze-out times by

$$\tau_\parallel(\hat{\epsilon}_\parallel) = \frac{|\hat{t}_\parallel|}{a}, \qquad \tau_\perp(\hat{\epsilon}_\perp) = \frac{|\hat{t}_\perp|}{a}, \qquad \hat{\epsilon}_{\parallel,\perp} = \left|\frac{\hat{t}_{\parallel,\perp}}{\tau_Q}\right|^a. \tag{15.50}$$

Solving (15.50) with (15.48) gives the anisotropic KZ times

$$|\hat{t}_\parallel| = (a\,\tau_{0\parallel})^{\frac{1}{1+a\nu_\parallel z_\parallel}}\,\tau_Q^{\frac{a\nu_\parallel z_\parallel}{1+a\nu_\parallel z_\parallel}}, \qquad |\hat{t}_\perp| = (a\,\tau_{0\perp})^{\frac{1}{1+a\nu_\perp z_\perp}}\,\tau_Q^{\frac{a\nu_\perp z_\perp}{1+a\nu_\perp z_\perp}}. \tag{15.51}$$

The corresponding freeze-out control parameters are

$$\hat{\epsilon}_\parallel = \left(\frac{a\,\tau_{0\parallel}}{\tau_Q}\right)^{\frac{a}{1+a\nu_\parallel z_\parallel}}, \qquad \hat{\epsilon}_\perp = \left(\frac{a\,\tau_{0\perp}}{\tau_Q}\right)^{\frac{a}{1+a\nu_\perp z_\perp}}. \tag{15.52}$$

Therefore the *directional* KZ lengths are

$$\boxed{\hat{\xi}_\parallel = \xi_{0\parallel}\left(\frac{\tau_Q}{a\,\tau_{0\parallel}}\right)^{\frac{a\nu_\parallel}{1+a\nu_\parallel z_\parallel}}, \qquad \hat{\xi}_\perp = \xi_{0\perp}\left(\frac{\tau_Q}{a\,\tau_{0\perp}}\right)^{\frac{a\nu_\perp}{1+a\nu_\perp z_\perp}}} \tag{15.53}$$

which reduce to the standard isotropic KZ law $\hat{\xi} \sim \xi_0(\tau_Q/\tau_0)^{\nu/(1+\nu z)}$ when $\nu_\parallel = \nu_\perp \equiv \nu$, $z_\parallel = z_\perp \equiv z$.

Principal-symbol estimate for (ν, z). From the linear operator in Vol. II, Eq. (V2-1.13), with dominant diffusive principal symbol $\lambda(\mathbf{k}) \simeq -D_\parallel k_\parallel^2 - D_\perp k_\perp^2 + m^2(\epsilon)$, one obtains the mean-field static exponent $\nu = 1/2$ via $m^2 \propto \epsilon$, and $z = 2$ from the diffusive poles. Anisotropy arises through $D_\parallel \neq D_\perp$ and possible direction-dependent mass renormalization, which feed into $\nu_{\parallel,\perp}$ at beyond-mean-field order.

Causal bound. The KZ length cannot exceed the (anisotropic) sound horizon at $|\hat{t}|$:

$$\hat{\xi}_\parallel \leq \frac{c_\parallel(\hat{t})}{H(\hat{t})}, \qquad \hat{\xi}_\perp \leq \frac{c_\perp(\hat{t})}{H(\hat{t})}. \tag{15.54}$$

This bound trims the scaling regime for extremely slow quenches ($\tau_Q \to \infty$) in an expanding background.

Inhomogeneous quenches. If the control parameter switches with a front speed v_f along $\hat{n}$, the relevant rate $|\epsilon/\dot{\epsilon}|$ is replaced by a local "crossing" time $\ell_{\hat{n}}/v_f$, and the KZ length saturates at $\hat{\xi}_{\hat{n}} \sim \min\{(15.53), \sqrt{D_{\hat{n}}\, \ell_{\hat{n}}/v_f}\}$. This case is relevant for bubble-like ignition patterns in Sec. **??**.

Defect Density Predictions and Uncertainty Budgets

Codimension counting with anisotropy. In $d = 3$, the density of topological defects scales with the *transverse* KZ lengths to the defect worldvolume:

$$\text{Domain walls (codim 1):} \quad \Sigma_{\text{wall}} \sim \hat{\xi}_{\perp,\text{wall}}^{-1}, \tag{15.55}$$

$$\text{Strings/vortices (codim 2):} \quad \mathcal{L}_{\text{str}}/V \sim \left(\hat{\xi}_\perp^{(1)}\, \hat{\xi}_\perp^{(2)}\right)^{-1}, \tag{15.56}$$

$$\text{Monopoles (codim 3):} \quad n_{\text{mon}} \sim \left(\hat{\xi}_\parallel\, \hat{\xi}_\perp^2\right)^{-1}, \tag{15.57}$$

where $\hat{\xi}_\perp^{(1)}, \hat{\xi}_\perp^{(2)}$ are the two transverse correlation lengths to a given string direction. If strings preferentially align with $\hat{\chi}$, then $\hat{\xi}_\perp^{(1)} = \hat{\xi}_\perp^{(2)} = \hat{\xi}_\perp$ and $\mathcal{L}_{\text{str}}/V \sim \hat{\xi}_\perp^{-2}$. For a random orientation distribution, an orientation average yields

$$\left\langle \mathcal{L}_{\text{str}}/V \right\rangle = \frac{1}{4\pi} \int d\Omega_{\hat{\ell}}\, \frac{1}{\hat{\xi}_\perp^2(\hat{\ell})}, \qquad \hat{\xi}_\perp^2(\hat{\ell}) = \hat{\xi}_\perp^2\,(1-\mu^2) + \hat{\xi}_\parallel\, \hat{\xi}_\perp\, \mu^2, \quad \mu \equiv \hat{\ell}\cdot\hat{\chi}, \tag{15.58}$$

which is bounded by $1/\hat{\xi}_\parallel \hat{\xi}_\perp \leq \langle \mathcal{L}/V \rangle \leq 1/\hat{\xi}_\perp^2$. Non-topological Hopf/Skyrme textures inherit an effective codimension 3 in the scaling estimate $n_{\text{hopf}} \sim (\hat{\xi}_\parallel \hat{\xi}_\perp^2)^{-1}$, up to model-dependent prefactors from the quartic terms (Vol. I, Ch. 4).

Putting the scalings together. Using (15.53), the *parametric* dependence of string density on the quench rate reads (for aligned strings)

$$\boxed{\frac{\mathcal{L}_{\text{str}}}{V} \sim \frac{1}{\xi_{0\perp}^2} \left(\frac{a\,\tau_{0\perp}}{\tau_Q}\right)^{\frac{2a\nu_\perp}{1+a\nu_\perp z_\perp}} \times\ \Theta_{\text{causal}}\,,} \tag{15.59}$$

with $\Theta_{\text{causal}} \in (0, 1]$ encoding the saturation due to the causal bound (15.54). Analogous expressions follow for walls and monopoles by replacing the appropriate product of $\hat{\xi}$'s.

Horizon and expansion effects. In an expanding background with Hubble rate H, the *physical* KZ lengths are $a(\hat{t})\hat{\xi}$. If $a(\hat{t})\hat{\xi} \gtrsim c/H$, defects are diluted and long-range correlations are washed out; practical production then obeys

$$\hat{\xi}_{\text{eff}} = \min\left\{\hat{\xi},\ \frac{c_s}{H}\right\}, \qquad (\mathcal{L}/V)_{\text{eff}} \to (\mathcal{L}/V)\big|_{\hat{\xi}\to\hat{\xi}_{\text{eff}}}. \tag{15.60}$$

Uncertainty budget and error propagation. Let Y be any density proxy scaling as a product $\prod_i \hat{\xi}_i^{-p_i}$ with $p_i > 0$ and $\hat{\xi}_i$ given by (15.53). Then

$$\ln Y = \ln Y_0 \; - \; \sum_i p_i \frac{a\nu_i}{1 + a\nu_i z_i} \; \ln\!\left(\frac{\tau_Q}{a\,\tau_{0i}}\right), \tag{15.61}$$

and the variance (to linear order) is

$$\begin{aligned}\mathrm{Var}[\ln Y] \simeq & \sum_i \left(\frac{p_i a\nu_i}{1 + a\nu_i z_i}\right)^2 \mathrm{Var}[\ln \tau_Q] + \sum_i \left(\frac{p_i a\nu_i}{1 + a\nu_i z_i}\right)^2 \mathrm{Var}[\ln \tau_{0i}] \\ & + \sum_i \left(\frac{p_i a}{(1 + a\nu_i z_i)^2} \ln\frac{\tau_Q}{a\,\tau_{0i}}\right)^2 \left(z_i^2\,\mathrm{Var}[\nu_i] + \nu_i^2\,\mathrm{Var}[z_i]\right) \; + \; \sigma^2_{\mathrm{cosmo}},\end{aligned} \tag{15.62}$$

where $\sigma^2_{\mathrm{cosmo}}$ aggregates cosmological sources (finite horizon, expansion history, inhomogeneous quench fronts, noise). For reporting, we propagate to fractional errors, e.g. $\Delta Y/Y \simeq \sqrt{\mathrm{Var}[\ln Y]}$.

Coarsening and late-time evolution. After freeze-out, curvature-driven coarsening (or string intercommutation) can reduce defect densities:

$$Y(t) \; = \; Y(\hat{t})\,\mathcal{C}\!\left(\frac{t}{\hat{t}}\right), \qquad \mathcal{C}(x) \; \sim \; x^{-\alpha_{\mathrm{dyn}}} \quad \text{for} \;\; x \gg 1, \tag{15.63}$$

with a dynamic exponent α_{dyn} depending on the universality class and the presence of expansion/friction. In cosmological applications we bracket predictions with $\alpha_{\mathrm{dyn}} \in [0, \alpha_{\max}]$ and treat $\mathcal{C}$ as a calibrated nuisance (Sec. 5.3).

Practical recipe (summary).

1. Choose $(\nu_\parallel, \nu_\perp, z_\parallel, z_\perp)$ from the linear principal symbol (mean-field) or from nonperturbative calibration (Vol. II, Ch. 11).
2. Determine the quench protocol (a, τ_Q) from the background η-wave dynamics (Ch. **??**).
3. Compute $\hat{\xi}_{\parallel,\perp}$ via (15.53), apply the causal caps (15.54)–(15.60).
4. Predict Y (walls/strings/monopoles) using (15.55)–(15.57) or the orientation-averaged form (15.58).
5. Attach uncertainties using (15.62) and, if needed, a coarsening factor $\mathcal{C}$ from numerical protocols (Ch. 11).

End-to-end case stub (to be detailed in Ch. 11). For a linear quench ($a = 1$), mean-field $\nu_\parallel = \nu_\perp = \frac{1}{2}$, $z_\parallel = z_\perp = 2$, and aligned strings, one finds

$$\hat{\xi}_\perp = \xi_{0\perp}\left(\frac{\tau_Q}{\tau_{0\perp}}\right)^{\frac{1}{4}}, \qquad \frac{\mathcal{L}_{\mathrm{str}}}{V} \sim \frac{1}{\xi^2_{0\perp}}\left(\frac{\tau_{0\perp}}{\tau_Q}\right)^{\frac{1}{2}},$$

modulo horizon capping and coarsening. This serves as a baseline for the synthetic-map forecasts in Part **??**.

Appendix D: Spectral Lattice and FFT Algorithms

Discretization, Boundary Conditions, and Stability

Fields and notation. We evolve the spectral aether fields $S = \{\rho, \eta, \alpha, \chi_I, \Phi, e^a{}_\mu\}$ on a comoving, periodic box of side L with $N_x \times N_y \times N_z$ collocation points. Let $a(t)$ denote the (possibly isotropized) emergent scale factor, and $\hat{\chi}$ the mean internal director that fixes preferred axes for anisotropic operators. We write spatial coordinates $\mathbf{x} \in [0, L)^3$ and Fourier wavenumbers $\mathbf{k} = (2\pi/L)\,(n_x, n_y, n_z)$, $n_i \in \{-\frac{N_i}{2}, \dots, \frac{N_i}{2} - 1\}$. Directional projections are $k_\parallel = \mathbf{k}\cdot\hat{\chi}$, $\quad \mathbf{k}_\perp = \mathbf{k} - k_\parallel \hat{\chi}$.

Pseudo-spectral differentiation and filtering. For any scalar f, the gradient/divergence/Laplacian consistent with the effective principal symbol (Vol. II, Sec. **??**) are implemented in Fourier space by multipliers

$$\widehat{\nabla_\parallel f} = i\,k_\parallel\,\hat{f}, \qquad \widehat{\nabla_\perp f} = i\,\mathbf{k}_\perp\,\hat{f}, \tag{15.64}$$

$$\widehat{\Delta_{\mathrm{ani}} f} = -\big(D_\parallel k_\parallel^2 + D_\perp |\mathbf{k}_\perp|^2\big)\,\hat{f}, \tag{15.65}$$

where $D_{\parallel,\perp}$ are direction-dependent diffusion/elastic coefficients evaluated on the background state. To control aliasing and Gibbs oscillations for nonlinearity $\mathcal{N}[S]$ we use a composite de-aliasing filter

$$\mathcal{P}_{2/3} \circ \mathcal{F}_p(\mathbf{k}), \qquad \mathcal{F}_p(\mathbf{k}) = \exp\Big[-\alpha\,(\|\mathbf{k}\|/k_{\max})^p\Big], \quad p \in \{8, 12\}, \quad \alpha \in [10^{-3}, 10^{-2}], \tag{15.66}$$

with the Orszag 2/3 truncation in each dimension; parameters are set by UQ runs (Sec. 11.1).

Time stepping (IMEX/ETD). Stiff linear parts (diffusive/elastic) are advanced with exponential integrators; nonlinear conservative parts are advanced explicitly:

$$\partial_t \mathbf{U} = \underbrace{\mathsf{L}\,\mathbf{U}}_{\text{stiff linear}} + \underbrace{\mathbf{N}(\mathbf{U}, t)}_{\text{nonlinear}}, \qquad \mathbf{U} \equiv \mathrm{stack}(S, \nabla S, \dots)\,, \tag{15.67}$$

$$\mathbf{U}^{n+1} = e^{\Delta t\,\mathsf{L}}\mathbf{U}^n + \sum_{m=0}^{M-1} \varphi_{m+1}(\Delta t\,\mathsf{L})\,\mathcal{B}_m[\mathbf{N}]\,, \quad \varphi_\ell(z) = \int_0^1 e^{(1-\theta)z}\,\frac{\theta^{\ell-1}}{(\ell-1)!}\,d\theta, \tag{15.68}$$

using ETDRK4 for smooth sectors and IMEX–SSP(2,2,2) when Rayleigh terms $\Gamma_\eta u^\mu \nabla_\mu$ are large. The emergent Hubble $H(t) = \frac{1}{3}\nabla_\mu u^\mu$ is either prescribed from a background solve or updated consistently by quadrature (Vol. II, Eq. (**??**)).

CFL and stability conditions. Let $c_{\max}$ be the largest effective signal speed among scalar/tensor sectors and $D_{\max} = \max(D_\parallel, D_\perp)$. For explicit parts, choose

$$\Delta t \;\le\; \min\left[\frac{\mathrm{CFL}_1}{c_{\max}}\,\min_i \Delta x_i,\;\; \frac{\mathrm{CFL}_2}{D_{\max}}\,\min_i \Delta x_i^2\right], \qquad \mathrm{CFL}_{1,2} \in (0.2, 0.5), \tag{15.69}$$

and for ETD/IMEX monitor the amplification factor $\max_{\mathbf{k}} \|\mathcal{A}(\Delta t, \mathbf{k})\| \le 1 + \varepsilon_{\mathrm{amp}}$ with $\varepsilon_{\mathrm{amp}} \sim 10^{-3}$. Energy and entropy diagnostics (Vol. II, Eq. (2.8)) are accumulated to ensure nonnegative production when $\eta > 0$.

Boundary conditions. We adopt periodic boundaries for cosmological boxes. For localized test problems (e.g. defect cores) we use sponge layers of thickness w with multiplicative mask $M(r) = \exp\left[-\beta(r - r_0)^2\right]$ or a perfectly-matched layer (PML) implemented by complex coordinate stretching in Fourier space: $k_i \to k_i - i\,\sigma_i(\mathbf{x})$, with σ_i tapered to zero in the interior.

Metric and anisotropy coupling. When evolving perturbations on a rolling background, the physical wavenumber scales as $k_{\rm phys} = k/a(t)$. Directional sound speeds $c_{\parallel,\perp}(t)$ enter as $\omega^2 = c_\parallel^2 k_\parallel^2/a^2 + c_\perp^2|\mathbf{k}_\perp|^2/a^2 + m_{\rm eff}^2$. Tensor modes use $\Delta t \le {\rm CFL}_T \min_i \Delta x_i/c_T$ with $|c_T - 1| \ll 1$ (Vol. II, Sec. 4.2).

Parallel FFTs and domain decomposition. We use pencil decomposition (2D slabs) with MPI for 3D FFTs, or vendor libraries (FFTW-MPI, P3DFFT). Strong scaling is validated by measuring wall-time per DoF and transpose overhead; weak scaling by fixing local $N_x \times N_y \times N_z$.

Reproducibility & checkpoints. All runs store: code commit hash, compiler flags, precision, RNG seeds, filter parameters, and gate tolerances. Checkpoints include $\widehat{S}(\mathbf{k})$ to avoid interpolation drift.

Defect Finders, Skeleton Extractors, and Validation Suites

Phase-based string/vortex finder (codimension 2). For a complex order proxy $\Psi = \rho^{1/2}e^{i\alpha}$ (or a suitable component of S), compute phase differences on each elementary plaquette (i, j) of a cell, wrapped to $(-\pi, \pi]$:

$$\Delta\alpha_{ij} = \mathrm{wrap}\big(\alpha_j - \alpha_i\big), \qquad W_\square = \frac{1}{2\pi}\,\mathrm{wrap}\left(\sum_\square \Delta\alpha\right) \in \mathbb{Z}. \tag{15.70}$$

A nonzero $W_\square$ marks a vortex line piercing the plaquette. Lines are constructed by connecting pierced plaquettes across adjacent cells; the total length density is $\mathcal{L}/V$ and can be orientation-resolved relative to $\hat{\chi}$.

Domain-wall finder (codimension 1). Given a scalar order parameter $\Theta(\mathbf{x})$ with two minima $\Theta_\pm$, walls are identified by zero-crossings of $F(\mathbf{x}) = \Theta - \Theta_*$ with Θ_* a mid-value between $\Theta_\pm$. Use marching-cubes to extract triangulated surfaces and accumulate area density $\Sigma_{\rm wall} = {\rm Area}/V$.

Monopoles & hedgehogs (codimension 3). For a unit vector field $\mathbf{n}(\mathbf{x})$ (e.g. normalized director), the discrete topological charge in a cell C is

$$Q_C = \frac{1}{8\pi}\sum_{\text{faces } f\subset\partial C} \Omega_f, \qquad \Omega_f = \arg\det\big[\mathbf{n}_{f,1}, \mathbf{n}_{f,2}, \mathbf{n}_{f,3}\big], \tag{15.71}$$

with Ω_f the oriented solid angle spanned by $\mathbf{n}$ at the triangle vertices of face f.

Hopfion invariant (linking of preimages). For $\mathbf{n} : \mathbb{T}^3 \to S^2$, compute the 2-form $F_{ij} = \mathbf{n}\cdot(\partial_i\mathbf{n} \times \partial_j\mathbf{n})$, solve $\nabla\cdot\mathbf{A} = 0,\ \nabla\times\mathbf{A} = \mathbf{F}$ (FFT-based Poisson solver), and evaluate the lattice Chern–Simons integral

$$Q_H = \frac{1}{32\pi^2}\sum_{\mathbf{x}} \epsilon^{ijk} A_i F_{jk}\,\Delta V, \tag{15.72}$$

which is gauge-invariant up to $\mathcal{O}(\Delta x^2)$ and robust under mild filtering.

Large-scale-structure skeleton and morphology. Given a smoothed intensity field $\mathcal{I}(\mathbf{x})$ (Sec. 7.1), compute the Hessian $H_{ij} = \partial_i \partial_j \mathcal{I}$ spectrally. Classify voxels by the sign pattern of eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$:

$$\begin{array}{lcl} \lambda_1, \lambda_2, \lambda_3 < 0: & \text{node/peak}, & \text{(cluster-like)} \\ \lambda_1, \lambda_2 < 0 < \lambda_3: & \text{filament}, & \text{(1D ridge)} \\ \lambda_1 < 0 < \lambda_2, \lambda_3: & \text{wall}, & \text{(2D sheet)} \\ 0 < \lambda_1, \lambda_2, \lambda_3: & \text{void/minimum}. & \end{array}$$

Extract the filamentary network by tracing integral curves along $\mathbf{e}_1$ (eigenvector of λ_1), with persistence thresholds set by topological persistence (Morse–Smale complex) to suppress noise.

Minkowski functionals and Betti numbers. For excursion sets $E_\nu = \{\mathcal{I} > \nu\}$, compute in 3D: $V_0 = \mathrm{Vol}(E_\nu),\ V_1 = \frac{1}{6}\mathrm{Area}(\partial E_\nu),\ V_2 = \frac{1}{6\pi}\int_{\partial E_\nu} H\, dA,\ V_3 = \frac{1}{4\pi}\int_{\partial E_\nu} K\, dA$ (mean and Gaussian curvature H, K on triangulated isosurfaces). Persistent homology yields Betti numbers $\beta_0, \beta_1, \beta_2$ for topology-aware validation.

Validation suites.

- **Spectral accuracy**: Manufactured solutions with known derivatives; measure L^2 error vs. N and verify exponential (spectral) convergence until limited by filter or aliasing.
- **Dispersion & dissipation**: Plane-wave tests for anisotropic speeds $c_{\parallel,\perp}$; compare numerical $\omega(\mathbf{k})$ vs. analytic dispersion.
- **Energy/entropy checks**: Track $\dot{\mathcal{E}}$ and $\sigma_{\mathrm{diss}} \geq 0$ for closed/open subsystems (Vol. II, Eq. (2.8)).
- **Topological robustness**: Inject synthetic defects with known charges (n, Q_H), advect and reconnect; verify charge conservation modulo reconnection rules.
- **Skeleton reliability**: Apply to Gaussian random fields with known analytic Minkowski functionals; compare recovered curves $V_i(\nu)$ and Betti diagrams.
- **Reproducibility & UQ**: Ensemble runs with randomized phases/seeds; report mean $\pm$ standard error for $(\mathcal{L}/V, \Sigma_{\mathrm{wall}}, n_{\mathrm{mon}}, Q_H)$ and morphology diagnostics.

Pseudocode (high-level loop).

One time step $t \to t + \Delta t$

1. Compute FFTs of all fields $\widehat{S}(\mathbf{k})$.
2. Apply linear stiff update via ETD/IMEX: $\widehat{S} \leftarrow e^{\Delta t\, \mathsf{L}(\mathbf{k})}\widehat{S} + \sum_m \varphi_{m+1}\, \widehat{\mathcal{B}}_m$.
3. Inverse FFT to real space; evaluate nonlinear terms $\mathcal{N}[S]$.
4. De-alias: apply $\mathcal{P}_{2/3} \circ \mathcal{F}_p$ to $\widehat{\mathcal{N}}$.
5. Complete explicit RK/ETD update; update $H(t)$, $M_{\mathrm{eff}}(\rho)$ if dynamic.
6. Every n_{diag} steps:
   - CFL/amp checks (15.69); energy/entropy diagnostics.
   - Defect finders (winding, walls, monopoles, Hopfion Q_H).
   - LSS skeleton, Minkowski functionals, Betti numbers.
7. Checkpoint fields and metadata if Δt_{chk} elapsed.

Performance notes. Use real-to-complex FFTs where applicable; fuse pointwise kernels; exploit SIMD for spectral multipliers; overlap MPI transposes with local computations; pin memory to reduce NUMA penalties. Mixed precision can accelerate non-stiff kernels while preserving diagnostics in double precision.

Appendix E: Transfer Functions and Boltzmann Hierarchies

Modifications to Scalar/Tensor Propagation

Metric, perturbation variables, and anisotropy. Perturb the emergent effective metric $g^{\rm eff}_{\mu\nu}$ in Newtonian gauge,

$$ds^2 \;=\; a^2(\eta)\left[-(1+2\Psi)\,d\eta^2 + (1-2\Phi)\,\gamma_{ij}\,dx^i dx^j\right], \quad \gamma_{ij} \simeq \delta_{ij}, \tag{15.73}$$

and decompose wavevectors relative to the internal director $\hat{\chi}$: $\mu \equiv \hat{\mathbf{k}}\cdot\hat{\chi}$, $k_\parallel = k\mu$, $\mathbf{k}_\perp = \mathbf{k} - k_\parallel\hat{\chi}$. Directional sound speeds from the spectral principal symbol (Vol. II, Sec. 4.1) are summarized by

$$c_s^2(\mu) \;=\; c_\perp^2 + (c_\parallel^2 - c_\perp^2)\,\mu^2 \equiv c_\perp^2 + \Delta c_s^2\,\mu^2, \qquad 0 < c_\perp^2 \le c_\parallel^2 \le 1, \tag{15.74}$$

with small anisotropy parameter $\epsilon_\chi \equiv \Delta c_s^2/c_\perp^2 \ll 1$ during inflation and the linear regime.

Modified Poisson and slip relations. The scalar constraints inherit state dependence through the running Planck mass $M_{\rm eff}(\rho)$ and possible spectral-curvature operators. We parameterize linear departures by (μ_S, γ_S):

$$k^2\Phi \;=\; 4\pi G a^2\,\mu_S(a,k,\mu)\,\rho\,\Delta, \qquad \Psi \;=\; \gamma_S(a,k,\mu)\,\Phi, \tag{15.75}$$
$$\mu_S \;=\; \frac{M_{\rm Pl}^2}{M_{\rm eff}^2}[1 + \delta\mu_0(a,k) + \delta\mu_2(a,k)\,\mathcal{P}_2(\mu) + \cdots], \quad \gamma_S \;=\; 1 + \delta\gamma_0(a,k) + \delta\gamma_2(a,k)\,\mathcal{P}_2(\mu) + \cdots.$$

Here $\mathcal{P}_\ell$ are Legendre polynomials; the μ-dependence encodes weak statistical anisotropy propagated from $\hat{\chi}$.

Running Planck mass and tensor friction. Define the running

$$\alpha_M(a) \;\equiv\; \frac{d\ln M_{\rm eff}^2}{d\ln a}. \tag{15.76}$$

Tensor (GW) modes h_{ij} in Fourier space satisfy[1]

$$h_{ij}'' + (2+\alpha_M)\,\mathcal{H}\,h_{ij}' + c_T^2\,k^2\,h_{ij} \;=\; 8\pi G\,a^2\,\Pi_{ij}^{(\rm T)}, \qquad c_T^2 \simeq 1 + \delta c_T, \tag{15.77}$$

with $\mathcal{H} \equiv a'/a$. The friction shift α_M and c_T map directly to spectral-curvature couplings (Vol. I, Ch. 3; Vol. II, Sec. 4.2).

Boltzmann hierarchies with directional sound speed. Photon temperature multipoles Θ_ℓ (scalar sector) obey

$$\Theta_0' = -k\,\Theta_1 - \Phi', \qquad \Theta_1' = \frac{k}{3}\,[\Theta_0 - 2\Theta_2] + k\,\Psi - \dot{\tau}\,(\Theta_1 - v_b), \tag{15.78}$$

[1] Anisotropic corrections $c_T(\mu) = 1 + \delta c_T^0 + \delta c_T^2\,\mathcal{P}_2(\mu) + \cdots$ are bounded by multi–messenger constraints; here we keep the leading small deviation.

$$\Theta'_\ell = \frac{k}{2\ell+1}\,[\ell\,\Theta_{\ell-1} - (\ell+1)\,\Theta_{\ell+1}] - \dot\tau\,\Theta_\ell \quad (\ell \geq 2), \tag{15.79}$$

where $\dot\tau = a n_e \sigma_T$. In the tightly–coupled baryon–photon fluid the Euler equation generalizes to

$$v'_b + \mathcal{H}\,v_b \;=\; -k\,\Psi + \frac{\dot\tau}{R}\,(\Theta_1 - v_b), \qquad c^2_{s,\gamma b}(\mu) \;=\; \frac{1}{3(1+R)}\,[1+\epsilon_\chi\,\mathcal{P}_2(\mu)]\,, \tag{15.80}$$

with $R \equiv 3\rho_b/4\rho_\gamma$. Free–streaming species (massless neutrinos, early Φ excitations) follow the standard collisionless hierarchy; anisotropy enters solely through the metric sources via (15.75).

E/B polarization and tensors. Polarization multipoles satisfy (schematically)

$$E'_\ell \;=\; \frac{k}{2\ell+1}[\ell\,E_{\ell-1} - (\ell+1)\,E_{\ell+1}] - \dot\tau\,(E_\ell - \sqrt{6}\,P\,\delta_{\ell 2}) + \cdots\,, \tag{15.81}$$

with the polarization source P built from Θ_2, E_2. Tensor sources include h' and inherit c_T, α_M through (15.77). Small anisotropy induces m-dependence in transfer functions (see §VIII).

Gauge restoration and consistency gates. Numerical implementations enforce: (i) subluminality $c_T^2 \leq 1$, $c_s^2(\mu) \leq 1$; (ii) positive $\dot\tau$ and entropy production (Vol. II, Eq. (V2-entropy)); (iii) restoration to $\mu_S \to M_{\rm Pl}^2/M_{\rm eff}^2$, $\gamma_S \to 1$, $c_T \to 1$ by recombination and later, within observational bounds.

Line-of-Sight Integration and Lensing Kernels

LOS solution with weak anisotropy. Temperature/polarization multipoles can be written as

$$a^X_{\ell m} \;=\; 4\pi\,(-i)^\ell \int \frac{d^3k}{(2\pi)^3} \sum_{\ell' m'} \mathcal{C}^{\ell m}_{\ell' m'}(\hat{\mathbf{k}}\cdot\hat\chi)\,\Delta^X_{\ell'}(k,\eta_0;\hat{\mathbf{k}}\cdot\hat\chi)\,Y^*_{\ell' m'}(\hat{\mathbf{k}})\,\mathcal{R}(\mathbf{k}), \tag{15.82}$$

where $\mathcal{R}$ is the primordial curvature mode and

$$\Delta^X_\ell(k,\eta_0;\mu) \;=\; \int_0^{\eta_0} d\eta\; S_X(\eta,k,\mu)\; j_\ell\Big(k\,\chi(\eta)\Big), \qquad \chi(\eta) = \eta_0 - \eta. \tag{15.83}$$

To first order in ϵ_χ we expand $S_X(\eta,k,\mu) = S_X^{(0)}(\eta,k) + \epsilon_\chi\,S_X^{(2)}(\eta,k)\,\mathcal{P}_2(\mu) + \cdots$ and $\mathcal{C}^{\ell m}_{\ell' m'} = \delta_{\ell\ell'}\delta_{mm'} + \mathcal{O}(\epsilon_\chi)$, so that statistical isotropy breaking appears as controlled (ℓ, m) mixing.

Scalar sources. The temperature source S_T decomposes into Sachs–Wolfe, Doppler, and ISW pieces modified by (μ_S, γ_S):

$$S_T^{(0)} = g(\eta)\;[\Theta_0 + \Psi] + \frac{d}{d\eta}\left[g(\eta)\frac{v_b}{k}\right] + e^{-\tau}\,(\Phi' + \Psi'), \tag{15.84}$$

$$S_T^{(2)} = g(\eta)\,\delta S_{\rm aniso}(\eta,k) + e^{-\tau}\,\delta(\Phi' + \Psi')\,, \tag{15.85}$$

with visibility $g(\eta) \equiv \dot\tau e^{-\tau}$ and $\delta S_{\rm aniso} \propto \epsilon_\chi$ arising from $c_s^2(\mu)$ in the tight–coupling oscillations and from μ–dependent μ_S, γ_S in (15.75).

Tensor sources. For $X = T, E, B$ the tensor parts involve h' and the polarization source,

$$S_X^{(\rm T)}(\eta,k) \;=\; e^{-\tau}\,\mathcal{S}_X[h',\,c_T,\,\alpha_M] + g(\eta)\,\mathcal{Q}_X[h], \tag{15.86}$$

with $\mathcal{S}_X, \mathcal{Q}_X$ the standard kernels, modified only through (c_T, α_M) in h evolution.

Weak lensing of CMB/LSS with modified Weyl response. Define the Weyl potential $\Psi_W \equiv (\Phi+\Psi)/2$ and its response

$$k^2\Psi_W \;=\; -4\pi G a^2\,\Sigma(a,k,\mu)\,\rho\,\Delta, \qquad \Sigma \;\equiv\; \tfrac{1}{2}\,\mu_S\,(1+\gamma_S). \tag{15.87}$$

The lensing potential along direction $\hat{\mathbf{n}}$ is

$$\phi(\hat{\mathbf{n}}) \;=\; -2\int_0^{\chi_*} d\chi\;\frac{\chi_*-\chi}{\chi_*\chi}\;\Psi_W\!\big(\chi\,\hat{\mathbf{n}},\eta\big)\,, \tag{15.88}$$

with last–scattering distance χ_*. In harmonic space, under Limber ($L \gtrsim 20$),

$$C_L^{\phi\phi} \;\simeq\; \int_0^{\chi_*}\frac{d\chi}{\chi^2}\left[\frac{2(\chi_*-\chi)}{\chi_*\chi}\right]^2 P_{\Psi_W}\!\left(k=\frac{L+1/2}{\chi},\,\mu=\hat{\mathbf{k}}\cdot\hat{\chi};\,\eta\right), \tag{15.89}$$

and P_{Ψ_W} inherits $\Sigma(a,k,\mu)$ from (15.87). Anisotropy induces M-dependence in $C_{LM}^{\phi\phi}$ and couplings that can generate TB/EB at $\mathcal{O}(\epsilon_\chi)$.

Galaxy lensing and cross–kernels. For a source distribution $n_s(\chi)$ normalized to unity, the convergence kernel is

$$W_\kappa(\chi) \;=\; \frac{3\,\Omega_{m0}\,H_0^2}{2}\,\frac{\chi}{a(\chi)}\int_\chi^\infty d\chi'\,n_s(\chi')\,\frac{\chi'-\chi}{\chi'}\;\times\;\Sigma\!\left(a,\frac{L+1/2}{\chi},\mu\right). \tag{15.90}$$

Cross–spectra $C_L^{\kappa T}$, $C_L^{\kappa E}$, and $C_L^{\kappa\,\mathrm{LSS}}$ follow by replacing one kernel with the corresponding LOS source (e.g. ISW for T).

Implementation notes (CAMB/CLASS–style).

- Replace (Φ,Ψ) solvers with (15.75), using $\mu_S(a,k,\mu)$, $\gamma_S(a,k,\mu)$; to linear order in ϵ_χ expand in $\mathcal{P}_2(\mu)$ and carry separate ℓm blocks or apply the Xi–matrix formalism for weak SI breaking.
- Evolve tensors with (15.77); expose (α_M, c_T) as background functions sourced by spectral couplings (Vol. I, Ch. 3).
- LOS sources S_X: add anisotropic pieces $S_X^{(2)}$ proportional to ϵ_χ and ensure the consistency gates (sub–luminality, restoration) are enforced at recombination and later.

Consistency gates (Appendix E).

1. **E1 Sub–luminality:** $c_T^2 \le 1$, $c_s^2(\mu) \le 1$ for all μ.
2. **E2 Positivity & stability:** no gradient/ghost instabilities in the photon–baryon fluid and free–streaming hierarchies.
3. **E3 Restoration:** $\mu_S \to M_{\rm Pl}^2/M_{\rm eff}^2$, $\gamma_S \to 1$, $c_T \to 1$ by recombination within observational limits.
4. **E4 SI bounds:** anisotropy coefficients $\propto \epsilon_\chi$ satisfy CMB g_* and TB/EB null tests.

Appendix F: GW Propagation and Multi-Messenger Constraints

Computing δc_T and Damping in Structured Media

Set–up and principal symbol. Tensor perturbations h_{ij} are defined as transverse–traceless (TT) fluctuations of the emergent metric $g^{\rm eff}_{\mu\nu}$ around a slowly varying background $(\mathcal{M}, g^{\rm eff})$ sourced by the spectral state $S = \{\rho, \eta, \Phi, \chi^\mu, T\}$. To leading order in gradients and anisotropy, the TT action reads

$$S_T^{(2)} = \frac{M_{\rm eff}^2}{8} \int d^4x \sqrt{-g^{\rm eff}} \left[\mathcal{G}^{\mu\nu\alpha\beta} \nabla_\mu h_{\alpha\beta} \nabla_\nu h_{\rho\sigma} \mathcal{I}^{\rho\sigma}{}_{\gamma\delta} - \mathcal{M}^2{}_{\gamma\delta}{}^{\alpha\beta} h_{\alpha\beta} h^{\gamma\delta} \right], \tag{15.91}$$

where $\mathcal{I}^{\rho\sigma}{}_{\gamma\delta}$ projects to TT, and the kinetic tensor $\mathcal{G}^{\mu\nu\alpha\beta}$ admits an isotropic GR limit $\mathcal{G}^{\mu\nu\alpha\beta} \to g^{\mu\nu}_{\rm eff} \mathcal{I}^{\alpha\beta}{}_{\gamma\delta}$. Spectral-curvature operators (Vol. I, Ch. 3) induce small deformations

$$\mathcal{G}^{\mu\nu\alpha\beta} \;=\; g^{\mu\nu}_{\rm eff} \mathcal{I}^{\alpha\beta}{}_{\gamma\delta} \;+\; \delta\mathcal{G}^{\mu\nu\alpha\beta}[\rho, \eta, \chi^\mu], \qquad \|\delta\mathcal{G}\| \ll 1, \tag{15.92}$$

and a negligible mass term $\mathcal{M}^2$ (we set $\mathcal{M}^2 = 0$ below).

In the aether rest frame u^μ and in WKB/TT gauge, the local dispersion relation for a plane wave $h \propto e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}$ is

$$\omega^2 \;=\; c_T^2(\hat{\mathbf{k}})\, k^2 \;-\; i\,\Gamma_T(\hat{\mathbf{k}})\,\omega \;+\; \mathcal{O}(\nabla\mathcal{G}), \tag{15.93}$$

where

$$c_T^2(\hat{\mathbf{k}}) \;=\; 1 + \delta c_T(\hat{\mathbf{k}}), \qquad \delta c_T(\hat{\mathbf{k}}) \;=\; \delta c_T^{(0)} \;+\; \delta c_T^{(2)}\, \mathcal{P}_2\!\big(\hat{\mathbf{k}}\cdot\hat{\chi}\big) + \cdots, \tag{15.94}$$

and Γ_T is an effective friction rate (see below). The anisotropy axis $\hat{\chi}$ is set by the internal director χ^μ (Vol. II, Sec. 4.2); $|\delta c_T^{(0,2)}| \ll 1$ by multi–messenger bounds.

Background (homogeneous) contribution. A running Planck mass $M_{\rm eff}(a)$ shifts the friction term through

$$h'' + (2 + \alpha_M)\,\mathcal{H}\,h' + c_T^2 k^2 h = 0, \qquad \alpha_M \equiv \frac{d\ln M_{\rm eff}^2}{d\ln a}, \tag{15.95}$$

with conformal time prime and $\mathcal{H} = a'/a$. The amplitude redshifts as

$$h(k,\eta) \;\propto\; \frac{1}{a(\eta)\, M_{\rm eff}(\eta)} \exp\!\left[-\tfrac{1}{2} \int_{\eta_e}^{\eta} d\tilde{\eta}\, \frac{\Gamma_T^{(\rm bg)}}{\mathcal{H}} \right], \qquad \Gamma_T^{(\rm bg)} \equiv 0 \text{ (pure running)}. \tag{15.96}$$

Thus, a background $\alpha_M \neq 0$ rescales the *GW luminosity distance*

$$d_L^{\rm GW}(z) \;=\; d_L^{\rm EM}(z) \exp\!\left[\frac{1}{2} \int_0^z d\tilde{z}\, \frac{\alpha_M(\tilde{z})}{1+\tilde{z}} \right]. \tag{15.97}$$

Structured–media (micro) damping and scattering. Spatial fluctuations of S (e.g. $\delta\rho, \delta\eta$, filamentary web; Sec. 7) induce stochastic refractive-index perturbations $\delta n_T(\mathbf{x}) \equiv -\frac{1}{2}\,\delta c_T(\mathbf{x})$ with correlation $C(\mathbf{r}) = \langle \delta n_T(\mathbf{x}) \delta n_T(\mathbf{x}+\mathbf{r}) \rangle$. In the Born approximation, the scattering mean free path is

$$\lambda_{\rm sc}^{-1}(k) \;=\; \frac{k^4}{4\pi} \int d^3r\, C(\mathbf{r})\, \frac{\sin(kr)}{kr}\,, \tag{15.98}$$

and the ensemble–averaged amplitude acquires an attenuation $\exp[-D/\lambda_{\rm sc}]$ over comoving distance D. The corresponding *effective* Γ_T in (15.93) is $\Gamma_T^{(\rm sc)} \simeq c_T/\lambda_{\rm sc}$. For wide–spectrum inhomogeneities, $\lambda_{\rm sc} \propto k^{-n}$ with $n \in [1,3]$ depending on the correlation spectrum (Kolmogorov–like vs. white).

Arrival–time and phase constraints. For a source at distance D and direction $\hat{\mathbf{n}}$, the group speed $v_g \simeq 1 + \delta c_T(\hat{\mathbf{n}})$ implies

$$\Delta t_{\mathrm{GW-EM}} \;\simeq\; -\,\delta c_T(\hat{\mathbf{n}})\,\frac{D}{c} \quad\Rightarrow\quad |\delta c_T^{(0)} + \delta c_T^{(2)}\mathcal{P}_2(\hat{\mathbf{n}}\cdot\hat{\chi})| \;\lesssim\; \frac{c\,|\Delta t|}{D}\,. \tag{15.99}$$

For chirps, propagation modifies the Fourier phase as

$$\Psi(f) \;=\; \Psi_{\mathrm{GR}}(f) \;-\; \pi\,\delta c_T^{\mathrm{eff}}\,D\,f \;+\; \cdots\,, \qquad \delta c_T^{\mathrm{eff}} \equiv \delta c_T^{(0)} + \delta c_T^{(2)}\mathcal{P}_2(\hat{\mathbf{n}}\cdot\hat{\chi}), \tag{15.100}$$

which is constrained by matched–filtering across ground/space–based bands.

Consistency gates (GW sector).

1. **F1 (subluminal cone):** $c_T^2(\hat{\mathbf{k}}) \le 1$ for all directions; $|\delta c_T^{(2)}| \ll 1$.
2. **F2 (weak scattering):** WKB validity $k\,\ell_{\mathrm{coh}} \gg 1$ and $D/\lambda_{\mathrm{sc}} \lesssim \mathcal{O}(1)$ for templates using geometric optics.
3. **F3 (luminosity consistency):** $d_L^{\mathrm{GW}}(z)$ from (15.97) consistent with standard sirens and EM distances within uncertainties.

Catalog-Level Likelihoods for PTA/LIGO/Virgo/KAGRA

Unified parameterization. We collect propagation parameters into

$$\Theta_{\mathrm{prop}} = \{\delta c_T^{(0)},\, \delta c_T^{(2)},\, \alpha_{M0},\, \nu,\, \kappa_{\mathrm{sc}}\},$$

where $\alpha_M(a) = \alpha_{M0}\,a^{\nu}$ controls d_L^{GW} via (15.97), and κ_{sc} sets $\lambda_{\mathrm{sc}}^{-1}(f) = \kappa_{\mathrm{sc}}\,f^n$ (spectral index n chosen per model). Source parameters are ϑ (masses, spins, sky, distance, inclination, ...).

Ground/space–based CBC likelihood (LIGO/Virgo/KAGRA/ET/CE). For a catalog $\mathcal{C} = \{d_i\}_{i=1}^{N_{\mathrm{CBC}}}$, the Gaussian noise likelihood is

$$\mathcal{L}_{\mathrm{CBC}}(\Theta_{\mathrm{prop}}, \{\vartheta_i\}) \propto \prod_{i=1}^{N_{\mathrm{CBC}}} \exp\!\left[-\tfrac{1}{2}\,\langle d_i - h_i(f;\vartheta_i,\Theta_{\mathrm{prop}}) \mid d_i - h_i(f;\vartheta_i,\Theta_{\mathrm{prop}})\rangle\right], \tag{15.101}$$

with inner product $\langle a \mid b\rangle = 4\,\mathrm{Re}\int_{f_{\mathrm{low}}}^{f_{\mathrm{high}}} df\; a^*(f)b(f)/S_n(f)$. Propagation enters via

$$\begin{aligned} h_i(f;\vartheta_i,\Theta_{\mathrm{prop}}) \;&=\; \frac{\mathcal{A}_i(f;\vartheta_i)}{d_{L,i}^{\mathrm{EM}}}\; \underbrace{\exp\!\left[-\tfrac{1}{2}\int_0^{z_i}\frac{\alpha_M(z)}{1+z}\,dz\right]}_{\equiv\,\mathcal{D}_M(z_i)}\; \underbrace{\exp\!\left[-\frac{D_i}{2\,\lambda_{\mathrm{sc}}(f)}\right]}_{\equiv\,\mathcal{D}_{\mathrm{sc}}(f,D_i)} \\ &\qquad\times \exp\!\Big\{i\big[\Psi_{\mathrm{GR}}(f;\vartheta_i) - \pi\,\delta c_T^{\mathrm{eff}}(\hat{\mathbf{n}}_i)\,D_i\,f\big]\Big\}, \end{aligned} \tag{15.102}$$

with $\delta c_T^{\mathrm{eff}}$ given by (15.100) and D_i the comoving distance.

Standard–siren sublikelihood (with EM counterparts). For events with redshift z (host identified or photometric), we use

$$\mathcal{L}_{\mathrm{SS}} \propto \prod_{j\in\mathrm{SS}} \exp\!\left[-\frac{\big(d_{L,j}^{\mathrm{GW}}(\Theta_{\mathrm{prop}}) - d_{L,j}^{\mathrm{EM}}\big)^2}{2(\sigma_{j,\mathrm{GW}}^2 + \sigma_{j,\mathrm{EM}}^2)}\right], \qquad d_L^{\mathrm{GW}} = d_L^{\mathrm{EM}}\,\mathcal{D}_M^{-1}. \tag{15.103}$$

PTA likelihood (continuous waves and stochastic background). For a pulsar timing array with timing residuals $\mathbf{r}$, the Gaussian-process likelihood is

$$\mathcal{L}_{\rm PTA}(\Theta_{\rm prop},\Theta_{\rm bg}) \propto \frac{\exp\left[-\frac{1}{2}\,\mathbf{r}^T\mathbf{C}^{-1}\mathbf{r}\right]}{\sqrt{\det(2\pi\mathbf{C})}}, \tag{15.104}$$

with covariance $\mathbf{C}=\mathbf{N}+\mathbf{S}$ (white/red noise $\mathbf{N}$, signal $\mathbf{S}$). For a stochastic GW background,

$$S_{ab}(f) \;=\; \Gamma_{ab}^{(\rm HD)}\,\frac{A_{\rm GW}^2}{12\pi^2}\left(\frac{f}{f_{\rm ref}}\right)^{-\gamma}\left[\frac{d_L^{\rm EM}}{d_L^{\rm GW}}\right]^2 \exp\!\left[-\frac{D_{\rm eff}}{\lambda_{\rm sc}(f)}\right], \tag{15.105}$$

where $\Gamma_{ab}^{(\rm HD)}$ is the Hellings–Downs overlap, and $D_{\rm eff}$ is an effective propagation distance that can incorporate anisotropy via $\langle \delta c_T^{(2)}\mathcal{P}_2\rangle$. For continuous waves (SMBHBs), phase tracking includes the linear-in-frequency propagation term $-\pi\delta c_T^{\rm eff} D f$.

Joint posterior and priors (gates as priors). The joint posterior for $\Theta_{\rm prop}$ and source/cosmological parameters Ξ is

$$\mathcal{P}(\Theta_{\rm prop},\Xi\mid\mathcal{D}) \propto \left[\mathcal{L}_{\rm CBC}\,\mathcal{L}_{\rm SS}\,\mathcal{L}_{\rm PTA}\right]\,\Pi_{\rm hard}(\Theta_{\rm prop})\,\Pi_{\rm soft}(\Theta_{\rm prop},\Xi), \tag{15.106}$$

where $\Pi_{\rm hard}$ enforces gates ($c_T\le 1$, small anisotropy $|\delta c_T^{(2)}|\ll 1$, α_M restoring by $z\sim 1100$), and $\Pi_{\rm soft}$ encodes external information (e.g. BBN/CMB).

Analytic null tests and summary constraints. Two robust, model–agnostic projections:

$$\text{Speed test:}\quad \delta c_T^{\rm sky} \;\equiv\; \arg\min_{\delta c_T^{(0)},\delta c_T^{(2)},\hat{\chi}} \sum_i \frac{\left[\Delta t_i + \delta c_T^{\rm eff}(\hat{\mathbf{n}}_i)\,D_i/c\right]^2}{\sigma_{t,i}^2}\,; \tag{15.107}$$

$$\text{Luminosity test:}\quad \alpha_{M0}^{\rm eff} \;\simeq\; \frac{2}{\ln(1+z)}\,\ln\!\left(\frac{d_L^{\rm GW}}{d_L^{\rm EM}}\right)\Bigg|_z\,, \tag{15.108}$$

binned in redshift and marginalized over weak lensing (magnification) and $\lambda_{\rm sc}$.

Forecast ingredients for multi–band catalogs. Sensitivity to $\delta c_T^{(0)}$ scales as $\sigma[\delta c_T^{(0)}]\sim c\,\sigma_t/D$ (prompt EM), to α_M as $\sigma[\alpha_M]\sim\sqrt{2}\,\sigma[\ln(d_L^{\rm GW}/d_L^{\rm EM})]/\ln(1+z)$, and to $\kappa_{\rm sc}$ through high–frequency SNR loss and PTA red spectrum curvature. Combining PTA (nHz) with LIGO band (10–10^3 Hz) breaks degeneracies in $(n,\kappa_{\rm sc})$.

Consistency gates (catalog level).

1. **C1 (template validity):** WKB/geometric–optics holds over each band used; if violated, include diffraction/lensing corrections.
2. **C2 (selection effects):** Propagation–induced SNR modulation accounted for in population likelihood to avoid Malmquist bias on $\alpha_M,\kappa_{\rm sc}$.
3. **C3 (multi–messenger coherence):** EM association probabilities enter $\mathcal{L}_{\rm SS}$ with marginalization over sky/host uncertainties.

Appendix G: Parameter Maps, Jacobians, and Priors

From Spectral Couplings to Observables

Spectral parameter vector. For compactness we collect the *fundamental* parameters of the spectral aether into

$$\Theta_{\rm spec} = \Big\{ V_0,\ \eta_\star,\ \kappa_\eta,\ \Gamma_\eta,\ M_{\rm eff},\ \delta c_T^{(0)},\ \delta c_T^{(2)},\ \hat{\chi},\ \mathcal{C}_\rho,\ \mathcal{C}_\chi,\ \dots \Big\}, \tag{15.109}$$

where $V_\eta(\eta) \sim V_0\, \mathcal{V}(\eta/\eta_\star)$, κ_η is the kinetic normalization, Γ_η the Rayleigh–type dissipation coefficient, $M_{\rm eff}$ the state–dependent Planck mass (evaluated on the inflationary background), $(\delta c_T^{(0)}, \delta c_T^{(2)}, \hat{\chi})$ encode the tensor sector speed and its quadrupolar anisotropy, and $\mathcal{C}_{\rho,\chi}$ schematically denote coefficients governing defect/self–interaction sectors (Chap. 5). The dots stand for any additional higher–derivative or curvature couplings retained in a given truncation.

Observable vector. Across the CMB/LSS/GW pipelines we assemble the *primary* observable summary into

$$\mathcal{O} = \Big\{ A_s,\ n_s,\ \alpha_s,\ r,\ c_T,\ f_{\rm NL}^{\rm (shape)},\ g_*,\ \Delta_{\rm LSS}(k,z),\ \mathcal{M}_{\rm morph},\ w_{\rm eff}(z),\ d_L^{\rm GW}(z),\ \delta c_T^{\rm sky}, \dots \Big\}, \tag{15.110}$$

where (A_s, n_s, α_s) are the scalar amplitude, tilt, and running at a pivot $k_\star$, r the tensor–to–scalar ratio, c_T the (angle–averaged) gravitational–wave phase speed, g_* a conventional measure of statistical anisotropy in the primordial spectrum, $f_{\rm NL}^{\rm (shape)}$ encodes non–Gaussianity amplitudes in designated templates, $\Delta_{\rm LSS}$ is the matter power spectrum (or transfer), $\mathcal{M}_{\rm morph}$ stands for morphological functionals (Minkowski functionals/skeleton) for the web, $w_{\rm eff}(z)$ the late–time effective EoS, $d_L^{\rm GW}(z)$ the GW luminosity distance (Appendix VIII), and $\delta c_T^{\rm sky}$ a sky–resolved speed estimator.

Map and Jacobians. The parameter–to–observable map is a deterministic composition

$$\mathcal{O} \;=\; \mathcal{F}\big(\Theta_{\rm spec}\big) = \underbrace{\mathcal{F}_{\rm late} \circ \mathcal{F}_{\rm trans} \circ \mathcal{F}_{\rm prim}}_{\text{late evolution}\circ\text{transfer}\circ\text{primordial}} \big(\Theta_{\rm spec}\big), \tag{15.111}$$

where:

- $\mathcal{F}_{\rm prim}$ produces $(A_s, n_s, \alpha_s, r, f_{\rm NL}, g_*, c_T)$ from the inflationary background and quadratic/cubic actions (Chaps. 4, App. VIII, VIII);
- $\mathcal{F}_{\rm trans}$ applies Boltzmann transfer and lensing kernels (App. VIII);
- $\mathcal{F}_{\rm late}$ evolves late–time background/perturbations, including residual η–relaxation and web morphology (Chaps. 8, 7).

Linear error propagation employs the total Jacobian

$$J \equiv \frac{\partial \mathcal{O}}{\partial \Theta_{\rm spec}} = \frac{\partial \mathcal{F}_{\rm late}}{\partial \mathcal{X}} \frac{\partial \mathcal{F}_{\rm trans}}{\partial \mathcal{Y}} \frac{\partial \mathcal{F}_{\rm prim}}{\partial \Theta_{\rm spec}}, \qquad \mathcal{Y} = \mathcal{F}_{\rm prim}(\Theta_{\rm spec}), \quad \mathcal{X} = \mathcal{F}_{\rm trans}(\mathcal{Y}). \tag{15.112}$$

Given a prior covariance $\mathbf{C}_\Theta$ on $\Theta_{\rm spec}$, the induced covariance on observables is

$$\mathbf{C}_{\mathcal{O}} \;\simeq\; J\, \mathbf{C}_\Theta\, J^{\mathsf{T}}. \tag{15.113}$$

Illustrative blocks of J. Using Sec. **??** and Ch. 4,

$$\frac{\partial \ln A_s}{\partial \ln V_0} = 1, \qquad \frac{\partial \ln A_s}{\partial \ln M_{\rm eff}} = -2, \qquad \frac{\partial \ln A_s}{\partial \ln \epsilon_\eta} = -1, \qquad \frac{\partial \ln A_s}{\partial \ln c_s} = -1; \tag{15.114}$$

$$\frac{\partial n_s}{\partial \epsilon_\eta} \simeq -2, \quad \frac{\partial n_s}{\partial \eta_\eta} \simeq -1, \quad \frac{\partial n_s}{\partial s_c} \simeq -1, \qquad r \simeq 16\,\epsilon_\eta\,\frac{c_s}{c_T} \;\Rightarrow\; \frac{\partial \ln r}{\partial \ln \epsilon_\eta} = 1, \; \frac{\partial \ln r}{\partial \ln c_s} = 1, \; \frac{\partial \ln r}{\partial \ln c_T} = -1. \tag{15.115}$$

Anisotropy enters through $c_s(\theta)$ and $c_T(\hat{\mathbf{k}})$:

$$\frac{\partial \ln \mathcal{P}_\zeta(k,\theta)}{\partial \delta c_s^{(2)}} = -\frac{1}{2}\,\mathcal{P}_2(\cos\theta), \qquad \frac{\partial \ln \mathcal{P}_T}{\partial \delta c_T^{(2)}} = -\frac{1}{2}\left\langle \mathcal{P}_2(\hat{\mathbf{k}}\cdot\hat{\chi})\right\rangle_{\rm sky}. \tag{15.116}$$

Worked example: $(V_0, \eta_\star) \mapsto (A_s, n_s, r)$. On a plateau potential $V_\eta = V_0\,[1 - e^{-\eta/\eta_\star}]^2$ with constant $(\kappa_\eta, M_{\rm eff}, \Gamma_\eta = 0)$,

$$\epsilon_\eta \simeq \frac{M_{\rm eff}^2 \kappa_\eta}{2}\left(\frac{V'_\eta}{V_\eta}\right)^2 = \frac{2M_{\rm eff}^2\kappa_\eta}{\eta_\star^2}\,\frac{e^{-2\eta/\eta_\star}}{(1-e^{-\eta/\eta_\star})^2}, \tag{15.117}$$

$$A_s \simeq \frac{V_\eta}{24\pi^2 M_{\rm eff}^4\,\epsilon_\eta\,c_s}, \qquad n_s - 1 \simeq -2\epsilon_\eta - \eta_\eta - s_c, \qquad r \simeq 16\,\epsilon_\eta\,\frac{c_s}{c_T}. \tag{15.118}$$

At the e–fold time $N(k_\star)$, $e^{-\eta/\eta_\star} \approx (1 + 2N\,\eta_\star^2/(M_{\rm eff}^2\kappa_\eta))^{-1}$, which closes the map and yields an analytic Jacobian for Fisher forecasts.

Late–time block: $(\alpha_M, \delta c_T, \kappa_{\rm sc}) \mapsto (d_L^{\rm GW}, \delta c_T^{\rm sky})$. Using Appendix VIII,

$$\frac{\partial \ln d_L^{\rm GW}}{\partial \alpha_{M0}} = \frac{1}{2}\,\frac{\partial}{\partial \alpha_{M0}}\int_0^z \frac{\alpha_M(z')}{1+z'}dz' = \frac{1}{2}\,\frac{z}{1+z} \;\; (\text{for } \alpha_M = \alpha_{M0}), \qquad \frac{\partial \delta c_T^{\rm sky}}{\partial \delta c_T^{(0)}} = 1, \; \frac{\partial \delta c_T^{\rm sky}}{\partial \delta c_T^{(2)}} = \mathcal{P}_2(\hat{\mathbf{n}}\cdot\hat{\chi}). \tag{15.119}$$

Numerical note. When composing J, we recommend automatic differentiation through the integrators of Chap. 11 to avoid step–size artifacts and to capture parameter dependence of k–to–N mapping.

Consistency-Gate Priors and Reparameterization Invariance

Gate taxonomy as priors. The *hard* consistency gates (Volume I and Chap. 12) translate into support constraints for the prior density $\Pi(\Theta_{\rm spec})$:

(S) Stability: $\mathsf{K} \succ 0$, $c_s^2(\theta) > 0$, $c_T^2(\hat{\mathbf{k}}) > 0$, $m_{\rm eff}^2$ above tachyon bounds,
(G) Gauge/Ward: Ward identities and soft limits satisfied to tolerance,
(TQ) Topological: defect charge quantization and allowed homotopy sectors,
(D) Dissipation/causality: $\eta_0 \geq 0$, $\sigma_{\rm diss} \geq 0$, no superluminal group speeds,
(N) Numerics: mesh/step convergence within prescribed thresholds.

We encode *hard* gates via indicator functions,

$$\Pi_{\rm hard}(\Theta_{\rm spec}) \;=\; \prod_a \mathbb{I}_a(\Theta_{\rm spec}), \qquad \mathbb{I}_a = \begin{cases} 1, & \text{if gate } a \text{ satisfied,} \\ 0, & \text{otherwise.} \end{cases} \tag{15.120}$$

Soft gates penalize near–violations, e.g. using barrier terms:

$$\Pi_{\rm soft}(\Theta_{\rm spec}) \;\propto\; \exp\Big[-\lambda_c\left\langle \max(0,\, c_s^2 - 1)\right\rangle - \lambda_h\,\|\min{\rm eig}(\mathsf{K})^{-1}\| - \lambda_{\rm aniso}\,(\delta c_T^{(2)})^2\Big], \tag{15.121}$$

with tunable Lagrange multipliers $(\lambda_c, \lambda_h, \lambda_{\rm aniso})$.

Reparameterization invariance. Let $\Theta' = \Phi(\Theta_{\rm spec})$ be a smooth bijection (e.g. switching to log–scales or principal components). For a prior density $\Pi(\Theta_{\rm spec})$, the transformed prior must obey

$$\Pi'(\Theta') \;=\; \Pi\big(\Phi^{-1}(\Theta')\big)\,\Big|\det\frac{\partial\Phi^{-1}}{\partial\Theta'}\Big|. \tag{15.122}$$

This guarantees that posteriors and evidences are invariant to reparameterization. In practice:

- For strictly positive scales ($V_0,\ \kappa_\eta,\ M_{\rm eff},\ \Gamma_\eta$), adopt *log–uniform* (Jeffreys) reference priors: $\Pi \propto 1/V_0$, etc., which are invariant under scale changes and non–informative over decades.
- For angular parameters (e.g. $\hat\chi$), use the rotationally invariant (Haar) measure on S^2: $d\Pi \propto d\Omega = \sin\theta\, d\theta\, d\phi$ and, for a *uniform* axis prior, $\cos\theta \sim \mathrm{Uniform}(-1,1)$, $\phi \sim \mathrm{Uniform}(0,2\pi)$.
- For anisotropy amplitudes, enforce smallness via Gaussian priors centered at 0 with width set by external bounds (e.g. $|\delta c_T^{(2)}| \lesssim 10^{-15}$ from multi–messenger).

Gate–aware reference priors. A convenient construction is a *product* prior in physically decoupled blocks, each truncated by its gates:

$$\Pi(\Theta_{\rm spec}) \;=\; \Pi_{\rm inf}(V_0,\eta_\star,\kappa_\eta,\Gamma_\eta,M_{\rm eff})\,\Pi_{\rm tens}(\delta c_T^{(0)},\delta c_T^{(2)},\hat\chi)\,\Pi_{\rm defect}(\mathcal{C}_\rho,\mathcal{C}_\chi)\;\cdots\;\times\;\Pi_{\rm hard}\,\Pi_{\rm soft}. \tag{15.123}$$

Example choices:

$$\Pi_{\rm inf} \propto \frac{1}{V_0}\,\frac{1}{\kappa_\eta}\,\frac{1}{M_{\rm eff}} \times \mathbb{I}\big(0<\epsilon_\eta<\epsilon_{\rm max},\ 0<c_s^2\le 1\big)\,, \tag{15.124}$$

$$\Pi_{\rm tens} \propto \mathcal{N}\Big(\delta c_T^{(0)};0,\sigma_0^2\Big)\,\mathcal{N}\Big(\delta c_T^{(2)};0,\sigma_2^2\Big)\,\frac{d\Omega}{4\pi}, \qquad \sigma_{0,2}\ \text{set by GW170817–like bounds.} \tag{15.125}$$

Jacobian–aware sampling. When sampling in derived variables (e.g. (A_s,n_s,r) instead of $(V_0,\eta_\star,\dots)$), include the absolute Jacobian determinant:

$$\Pi_{\rm pullback}(\Theta_{\rm spec}) \;=\; \Pi_{\rm obs}\big(\mathcal{F}(\Theta_{\rm spec})\big)\,\Big|\det\frac{\partial\mathcal{F}}{\partial\Theta_{\rm spec}}\Big|, \tag{15.126}$$

so that the implied prior in observable space matches the analyst's intent (e.g. flat in A_s and n_s).

Hierarchical and nuisance layers. Calibration parameters (beam, noise, photometric redshifts) and astrophysical nuisance terms (bias, baryonic feedback) can be appended as ϕ with independent priors $\Pi(\phi)$ and profiled or marginalized. To prevent leakage of gate–violations through nuisance directions, couple soft penalties to *derived* observable–level gate metrics (e.g. subluminality of fitted c_T curves).

Prior predictive checks and null–tests. Before confronting data, draw $(\Theta_{\rm spec}) \sim \Pi$ and produce $(\mathcal{O})$ via fast surrogates (Chap. 10). Reject priors that populate excluded regions (e.g. $r > 0.1$, $|g_*|$ above CMB limits) *without* fine tuning. This operationalizes the "gates as priors" philosophy and reduces posterior pathologies.

Summary. This appendix provides the calculus required to (i) propagate spectral parameters to observables with transparent Jacobians, (ii) encode theoretical gates as principled priors, and (iii) maintain reparameterization invariance across sampling choices. Together, these rules ensure that inference outcomes reflect *physics* rather than artifacts of coordinates or ill–posed prior volumes.

Appendix H: Data Likelihoods and Systematics Models

Foregrounds, Calibration, and Beam Systematics

Overview. This appendix formalizes the data models used across CMB, LSS, lensing and GW datasets, emphasizing *foreground emission*, *calibration and bandpass*, and *beam/point-spread* effects. All nuisance parameters enter the joint likelihood with *gate-aware priors* (cf. App. VIII), and are either profiled or marginalized analytically where possible.

CMB component separation in harmonic space

Let $\mathbf{d}_\nu(\hat{\mathbf{n}})$ denote the sky map at frequency ν (temperature or Q/U polarization). In the linear mixture model,

$$\mathbf{d}_\nu(\hat{\mathbf{n}}) = g_\nu \sum_{c\in\{\mathrm{CMB,\,sync,\,dust,\,AME,\,\dots}\}} A_{\nu c}(\boldsymbol{\theta}_c,\nu)\,\mathbf{s}_c(\hat{\mathbf{n}}) \;+\; \mathbf{n}_\nu(\hat{\mathbf{n}}), \tag{15.127}$$

where $g_\nu = 1+\varepsilon_\nu$ is the (multiplicative) calibration, $A_{\nu c}$ the mixing matrix, $\mathbf{s}_c$ component maps, and $\mathbf{n}_\nu$ instrument noise. For polarized foregrounds we allow decorrelation with spatially varying spectral parameters $\boldsymbol{\theta}_c(\hat{\mathbf{n}})$ controlled by low-rank templates.

In multipoles, including beams and masks,

$$\tilde{a}^X_{\ell m}(\nu) = B^X_\ell(\nu)\sum_c A_{\nu c}\,a^X_{\ell m}(c) \;+\; \sum_{\ell' m'} K^X_{\ell m,\ell' m'}(\nu)\,a^X_{\ell' m'}(\nu) \;+\; n^X_{\ell m}(\nu), \tag{15.128}$$

with $X\in\{T,E,B\}$, B^X_ℓ the isotropic beam transfer and K the (known) mask-induced mode-coupling.

Foreground SEDs (temperature). We adopt standard parametric forms, with amplitudes referenced to a pivot ν_0:

$$\text{Synchrotron: } I_{\mathrm{s}}(\nu) = A_{\mathrm{s}}\left(\frac{\nu}{\nu_0}\right)^{-\beta_{\mathrm{s}}}, \qquad \beta_{\mathrm{s}}>0, \tag{15.129}$$

$$\text{Thermal dust: } I_{\mathrm{d}}(\nu) = A_{\mathrm{d}}\left(\frac{\nu}{\nu_0}\right)^{\beta_{\mathrm{d}}}\frac{B_\nu(T_{\mathrm{d}})}{B_{\nu_0}(T_{\mathrm{d}})}, \qquad \beta_{\mathrm{d}}>0,\ T_{\mathrm{d}}\sim 18{-}25\ \mathrm{K}, \tag{15.130}$$

and an AME (spinning dust) template $I_{\mathrm{AME}}(\nu) = A_{\mathrm{ame}}\,\mathcal{S}_{\mathrm{ame}}(\nu;\nu_{\mathrm{p}},\Delta)$ peaked at $\sim$ 20–40 GHz. Polarized SEDs follow Eqs. (15.129)–(15.130) with separate amplitudes and allow a *decorrelation factor* $r_{\mathrm{dec}}(\nu_1,\nu_2)\le 1$:

$$\mathcal{C}^{XY,\mathrm{dust}}_\ell(\nu_1,\nu_2) = r_{\mathrm{dec}}(\nu_1,\nu_2)\sqrt{\mathcal{C}^{XY,\mathrm{dust}}_\ell(\nu_1,\nu_1)\,\mathcal{C}^{XY,\mathrm{dust}}_\ell(\nu_2,\nu_2)}. \tag{15.131}$$

Calibration and bandpass. Bandpass mismatch is modeled as small spectral distortions around the nominal response:

$$d_\nu \to g_\nu \int d\nu'\,\mathcal{R}_\nu(\nu')\,d(\nu') \simeq g_\nu\Big[d(\nu) + \sum_k \alpha_{\nu k}\,\partial^k_{\ln\nu} d(\nu)\Big], \tag{15.132}$$

with nuisance $\alpha_{\nu k}$ and Gaussian priors from lab calibrations. Absolute calibration uncertainty is encoded by ε_ν with a shared hyperprior or cross-calibrated via dipole/planetary references.

Beams and window functions. For mildly non-circular beams, we use an *effective* isotropic window B_ℓ augmented by a small anisotropic leakage kernel $L^{XY}_{\ell m}$ (coupling $T \to E/B$, $E \leftrightarrow B$) constrained by planet scans:

$$\mathbf{C}_b^{XY} \;=\; \sum_\ell W_{b\ell}\, B_\ell^X B_\ell^Y \left[\mathbf{M}^{XY}_{\ell\ell'}\, C^{XY}_{\ell'}\right] \;+\; \Delta_b^{XY}(\text{leakage}), \tag{15.133}$$

where $W_{b\ell}$ are bandpower windows and $\mathbf{M}$ the MASTER coupling matrix. The leakage term is linear in calibrated beam-harmonic coefficients with Gaussian priors.

Power-spectrum likelihood and cross-spectra

We adopt a Gaussian/Wishart hybrid likelihood for binned cross-spectra between (noise) splits $i \neq j$:

$$-2\ln \mathcal{L}_{\rm CMB} = \sum_b \Big(\hat{\mathbf{C}}_b - \mathbf{C}_b(\vartheta)\Big)^{\mathsf{T}} \boldsymbol{\Sigma}_b^{-1} \Big(\hat{\mathbf{C}}_b - \mathbf{C}_b(\vartheta)\Big), \tag{15.134}$$

with $\hat{\mathbf{C}}_b$ collecting $\{TT, TE, EE, BB\}$ cross bandpowers, model vector $\mathbf{C}_b(\vartheta)$ depending on cosmological and nuisance parameters, and covariance $\boldsymbol{\Sigma}_b$ including sample variance, noise, mask coupling, and foreground residuals from (15.127). Cross-only spectra suppress noise bias; residual split-correlated noise is modeled via small nuisance terms with half-Gaussian priors.

LSS and lensing systematics

The observed galaxy (or intensity-mapping) overdensity in redshift bin α is

$$\delta^{(\alpha)}_{\rm obs}(\mathbf{x}) = \big(1 + \gamma^{(\alpha)}_{\rm cal}\big)\, \big[\delta^{(\alpha)}_g(\mathbf{x}) + \sum_p \beta^{(\alpha)}_p\, T_p(\mathbf{x})\big] \;+\; \epsilon^{(\alpha)}(\mathbf{x}), \tag{15.135}$$

where $\gamma_{\rm cal}$ encodes multiplicative calibration (depth/seeing), $\{T_p\}$ are systematics templates (stellar density, extinction, airmass), β_p their coefficients, and ϵ noise. In Fourier space, the pseudo-power is

$$\hat{P}_{\rm obs}(k,\mu) = \big(1 + 2\gamma_{\rm cal}\big)\, P_s(k,\mu) + \sum_{pq} \beta_p \beta_q\, P_{pq}(k) + N, \tag{15.136}$$

where P_s includes bias, RSD, FoG and BAO damping; P_{pq} are template auto/cross terms. We perform *template deprojection* by augmenting the covariance with large prior variances on β_p, or regress them out at map level. For weak lensing, PSF leakage and shear calibration (m, c) enter the two-point functions:

$$\xi^{\pm}_{\rm obs}(\theta) = (1+m_i)(1+m_j)\, \xi^{\pm}_{ij}(\theta) + \Delta^{\pm}_{\rm PSF}(\theta) + \cdots\,, \tag{15.137}$$

with Gaussian priors on (m_i) from image simulations and PSF residual templates propagated into $\Delta_{\rm PSF}$.

GW propagation and calibration

For N GW events with strain calibration $\kappa_{\rm cal}$ and population selection $\mathcal{S}$, the hierarchical likelihood is

$$\mathcal{L}_{\rm GW} \propto \prod_{n=1}^{N} \int d\phi_n\, p\big(d_n \mid \phi_n, \kappa_{\rm cal}, \delta c_T(\hat{\mathbf{n}}_n), \dots\big)\, p\big(\phi_n \mid \vartheta, \mathcal{S}\big) \Big/ \alpha(\vartheta, \mathcal{S}), \tag{15.138}$$

where ϕ_n are intrinsic parameters; α is the selection factor. The GW luminosity distance is modified by running Planck mass (App. VIII):

$$d_L^{\rm GW}(z) = d_L^{\rm EM}(z)\, \exp\left[\tfrac{1}{2}\int_0^z \frac{\alpha_M(z')}{1+z'} dz'\right], \tag{15.139}$$

and the phase accumulates δc_T as a (bounded) dispersive correction; $\kappa_{\rm cal}$ absorbs strain-amplitude systematics with Gaussian priors per detector.

Priors and analytic marginalization

Where linear in amplitudes (e.g. foreground A_c, template coefficients β_p), we marginalize analytically:

$$\mathcal{L}_{\rm marg}(\vartheta) = \int d\mathbf{a}\, \mathcal{L}(\vartheta, \mathbf{a})\, \Pi(\mathbf{a}) \;\propto\; \frac{\exp\big[-\frac{1}{2}\mathbf{r}^{\mathsf{T}}\mathbf{C}_{\rm eff}^{-1}\mathbf{r}\big]}{\sqrt{\det \mathbf{C}_{\rm eff}}}, \tag{15.140}$$

with $\mathbf{a}$ the linear nuisances, residual vector $\mathbf{r}$ and an *effective* covariance $\mathbf{C}_{\rm eff}$ that includes prior-induced penalties. This stabilizes sampling and prevents foreground overfitting.

Quality gates. Each survey block exposes diagnostics (null spectra, beam window residuals, template leakage) that feed *soft* penalties in the joint prior (App. VIII), ensuring that posterior support concentrates on systematics-safe regions.

Posterior Predictive Checks and Null Tests

Posterior predictive scheme. Given posterior draws $\{\vartheta^{(s)}\}_{s=1}^{S}$ from the joint model (cosmology + nuisances), generate replicated datasets $D^{\rm rep,(s)}$ via the full forward model (including beams, masks, noise, selection). For each test statistic $T(\cdot)$,

$$p_{\rm PPC} \equiv \Pr\big[T(D^{\rm rep}) \geq T(D_{\rm obs}) \mid D_{\rm obs}\big] \approx \frac{1}{S}\sum_{s=1}^{S} \mathbb{I}\Big[T\big(D^{\rm rep,(s)}\big) \geq T(D_{\rm obs})\Big]. \tag{15.141}$$

We consider:

- **CMB:** split-difference spectra (HM1–HM2, DS1–DS2), frequency-difference maps, EB leakage spectra, dust residual cross–frequency decorrelation; χ^2 of null bandpowers vs. $\mathbf{\Sigma}_b$.
- **LSS:** systematics-template cross–correlations, depth/seeing binned power, North/South jackknifes, E/B-mode separation for shear, star density deprojection residuals.
- **GW:** sky- and distance–distribution residuals vs. selection, calibration-consistency across detectors, correlation of δc_T estimates with propagation environment.

Frequentist-style null tests (diagnostics). We deploy a battery of near-orthogonal nulls:

1. *Half-mission/half-ring* cross-spectra: $\hat{C}_\ell^{XY,\,\rm HM1\times HM2}$ should be consistent with the model minus noise bias; difference spectra must be null.
2. *Jackknife maps:* tile the survey into $N_{\rm jk}$ regions; compute variance of $\hat{C}_\ell$ over leave-one-out sets; compare to PPC bands.
3. *Template flips:* regress on T_p, then *flip* the template phase and verify deprojection; residual correlation coefficient $\rho \to 0$ within uncertainty.
4. *Anisotropy stress-test:* reconstruct g_* and quadrupole of $c_s(\theta)$ from ζ; verify that recovered values in *null* simulations are unbiased and consistent with posterior constraints.

Goodness of fit and tension metrics. We report χ^2 per bandpower block, Bayesian p-values, and parameter-shift metrics (e.g. $\Delta\chi^2$ upon freeing a nuisance block). Posterior predictive loss $L = \mathbb{E}[(D^{\rm rep} - D_{\rm obs})^{\mathsf{T}}\mathbf{W}(D^{\rm rep} - D_{\rm obs})]$ is monitored with weights $\mathbf{W}$ chosen to upweight known failure modes (e.g. high-ℓ polarization).

End-to-end (E2E) validation. We finalize with E2E simulations: inject a known spectral-cosmology signal through timestreams / scanning / mapmaking / component separation / bandpowers / inference. Recovery of $(n_s, r, f_{\rm NL}, g_*, c_T, w_{\rm eff})$ within forecasted uncertainties is the pass criterion; systematic residuals are required to lie within the *soft* gate tolerances (App. VIII). Failures trigger targeted model extensions (extra nuisance degrees, template enrichments) or data quality cuts documented alongside the pipeline.

Summary. The likelihoods and systematics models of this appendix provide a *closed*, gate-aware loop: parametric foregrounds with decorrelation, calibrated beams/bandpasses, cross-split spectra for unbiased estimation, LSS/lensing template control, and GW catalog-level hierarchies. Posterior predictive checks and structured null tests ensure that reported spectral-cosmology inferences are driven by *sky* rather than *systematics*.

Part IX

Conclusion to Volume II

Chapter 16

Synthesis, Limitations, and Outlook

Concise Assessment and Execution Plan

Bottom line. The four thrusts below target the highest scientific leverage: (1) near-term, falsifiable observables; (2) critical early–universe dynamics; (3) EFT consistency across scales; (4) a compact mathematical closure that binds the framework. Keep this ordering and deliver each thrust as a small, "submission-ready" package. For each thrust we state its value, risks, and the minimal viable paper (MVP).

1) Full $f_{\rm NL}$ file (cubic action + in–in + anisotropic templates)

Value. A distinctive, immediately testable signature (angle-dependent bispectrum) comparable to Planck/ACT/Simons; a clean discriminator from single-field isotropic templates.

Risks. Gauge choice and a clean separation of quadratic vs. cubic orders with angle-dependent sound speed $c_s(\theta)$; possible field-redefinition subtleties near limiting cases.

MVP.

- Quadratic action $S^{(2)}[\zeta]$ with controlled anisotropy

$$c_s(\theta) = c_0[1 + \epsilon\, P_2(\cos\theta)]\,,$$

  and cubic action $S^{(3)}$ through the operators $\dot\zeta^3$, $\dot\zeta(\partial\zeta)^2$, $\zeta(\partial\zeta)^2$ with angle-weighted Wilson coefficients (Legendre or Y_{2m} expansion).

- In–in computation of $B_\zeta(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ and extraction of templates: local, equilateral, orthogonal, plus a *directional* template tied to P_2.

- Ready-to-compare deliverables: a table mapping Wilson coefficients to ϵ; three shape-function plots; numerical targets for $f_{\rm NL}^{\rm loc,eq,ortho}$ and the directional $f_{\rm NL}^{(2)}$ with forecast bands.

2) Precision reheating: scanning Γ_η and energy-flow maps to $\Delta N_{\rm eff}$/BBN

Value. Direct linkage between the end of inflation, $T_{\rm RH}$, BBN bounds, and $\Delta N_{\rm eff}$; yields viability windows and sharp exclusions.

Risks. Transitions between perturbative and nonperturbative (Floquet) regimes; accurate energy bookkeeping with background expansion.

MVP.

- A compact parameter box: $\{\Gamma_\eta,\ \kappa_\eta,\ q_{\rm res},$ branching to Φ/structural$\}$.
- Three-fluid balance equations (η / Φ / radiation) to derive $T_{\rm RH}$ and the bound $T_{\rm RH} > 5\,\mathrm{MeV}$.
- Computation of
$$\Delta N_{\rm eff} = \frac{8}{7}\left(\frac{11}{4}\right)^{4/3}\frac{\rho_{\rm extra}}{\rho_\gamma}\,,$$
with energy-flow charts highlighting allowed / edge / excluded regions vs. BBN and CMB constraints.
- Side note: potential high-frequency GW features from preheating as an auxiliary prediction.

3) RG gates: phase/flow diagrams + hyperbolicity/causality

Value. EFT consistency across scales with a visible *safety region* for parameter selection; prevents superluminality, ghosts, and strong-coupling pathologies.

Risks. Defining the correct causal cone relative to $g_{\rm eff}$; locating any sharp front-velocity bounds for c_s, c_T if present.

MVP.

- Few, meaningful running variables: $Z_\perp,\ \Delta Z,\ M_{\rm eff},\ c_s^2,\ c_T^2$.
- **On-flow gates**:
  1. **No ghosts:** kinetic blocks >0.
  2. **No gradient instabilities:** $c_s^2 > 0,\ c_T^2 > 0$.
  3. **Causality/analyticity:** front velocities within the $g_{\rm eff}$ cone; forward-limit positivity for $(\partial\phi)^4$-type operators.
  4. **No strong coupling:** $\Lambda_{\rm sc} > \max(H,\ E_{\rm proc})$.
- Graphical output: a shaded phase/flow diagram marking the *admissible* region and labeled failure modes at the boundaries.

4) Appendix "Emergence of Laws": a compact prooflet

Value. Closes the philosophical–mathematical loop succinctly so the volume is self-contained.

Risks. Avoid overpromising; deliver a crisp, technically modest argument.

MVP.

- **(A) Hilbert space from $\mathcal{L}^{(2)}$:** extract the quadratic form and symplectic structure to define an inner product and quantize ζ/Φ modes.
- **(B) Restoration lemma across scales:** under RG + effective dissipation, larger-scale laws restore approximate symmetries (stabilized c_s, c_T; directional isotropy as $\eta\downarrow 0$).
- **(C) Measurement rules as effective limits with $\eta > 0$:** bound the operational energy/time of measurement by η-controlled decoherence scales (no extra postulates).

Suggested Order of Work

1. Execute **(1)** and **(2)** first to produce immediately testable numbers (CMB/BBN/$\Delta N_{\rm eff}$).
2. Build **(3)** to delimit the safe parameter region for (1)(2).
3. Conclude with **(4)** as a short appendix that secures foundations.

Deliverables Checklist

- Tables and `plots` for the bispectrum with an independent *directional* template.
- Maps $(\Gamma_\eta, \kappa_\eta) \mapsto \{T_{\rm RH}, \Delta N_{\rm eff}\}$ with BBN/CMB shading.
- An RG phase/flow diagram with the consistency gates overlaid.
- A three-page appendix containing the compact prooflet in (4).

16.1 What Was Gained: From Quiescent Void to Late-Time Acceleration

Arc of the volume. This volume developed a *closed* cosmological narrative within the Quantum–Aether framework:

$$\text{Quiescent Void } (S_0) \xrightarrow{\text{instability}} \eta\text{–wave inflation} \xrightarrow{\text{exit}} \text{hot epoch + seeds} \xrightarrow{\text{growth}} \text{web resonance} \xrightarrow{\text{relaxation}} w_{\rm eff}(t) \gtrsim -$$

Each stage was anchored in the *same* structural degrees of freedom introduced in Volume I, with quantitative outputs and gate-based consistency tests.

(Gains 1) Foundations and ignition. Chaps. **??**–**??** formalized the quiescent reference state S_0 and its linear response:

$$\mathcal{L}^{(2)} = \tfrac{1}{2}\,\delta\mathbf{S}^{\mathsf{T}}\left[\mathsf{K}\,\partial_t^2 + \mathsf{C}\,\partial_t + \mathsf{Z}^{ij}\nabla_i\nabla_j + \mathsf{M}^2\right]\delta\mathbf{S}, \qquad \lambda_\alpha(\mathbf{k}) = \lambda_{\alpha,0} - D_{\alpha,\parallel}k_\parallel^2 - D_{\alpha,\perp}k_\perp^2 + \dots \tag{16.1}$$

identifying modulational bands and anisotropy set by $\hat{\chi}^{(0)}$. The internal clock T and viscosity η supplied, respectively, *temporal order* and a *positive* entropy-production arrow.

(Gains 2) Inflation without inflaton. Chap. **??** showed that a coherent η-condensate drives quasi–de Sitter with slow parameters

$$\epsilon_\eta \simeq \frac{\kappa_\eta M_{\rm eff}^2}{2}\left(\frac{V_\eta'}{V_\eta}\right)^2\left(\frac{3H}{3H+\Gamma_\eta/\kappa_\eta}\right)^2, \qquad \eta_\eta \equiv \frac{\kappa_\eta M_{\rm eff}^2 V_\eta''}{V_\eta}, \tag{16.2}$$

and graceful exit via intrinsic Rayleigh dissipation Γ_η. No ad hoc inflaton sector is required.

(Gains 3) Primordial spectra with controlled anisotropy. Chap. 4 derived

$$\mathcal{P}_\zeta(k,\theta) = \frac{H^2}{8\pi^2\,M_{\rm eff}^2\,\epsilon_\eta\,c_s(\theta)}\bigg|_{k=aH/c_s}, \quad n_s - 1 \simeq -2\epsilon_\eta - \eta_\eta - s_c, \quad r \simeq 16\,\epsilon_\eta\,\frac{c_s}{c_T}, \tag{16.3}$$

with mild directional dependence through $c_s(\theta) = c_\perp + \Delta c\,\cos^2\theta$, and a bispectrum whose shapes are tied to η-self interactions and ζ-$\delta\chi$ mixing. Tensor speed deviations δc_T were kept within multi-messenger bounds.

(Gains 4) Defects/solitons as dark matter. Chap. 5 mapped Kibble–Zurek freeze-out in an anisotropic medium, predicted defect classes $(\pi_0, \pi_1, \pi_2, \pi_3)$, and established mass/self-interaction scalings $M_{\rm sol} \sim \rho_0\ell^3 + \mathcal{T}\,\ell$, with σ/M governed by core structure—linking cosmology to the soliton physics of Vol. I.

(Gains 5) Exit, thermal history, and initial conditions. Chap. 6 quantified Γ_η-driven reheating into Φ and structural modes, computed $T_{\rm rh}$, checked BBN/Neff bounds, and produced transfer functions carrying spectral imprints to LSS.

(Gains 6) Web morphology as resonance. Chap. 7 built a Chladni-like morphology from interference of unstable modes; beyond gravity statistics (Minkowski functionals, skeleton, bispectrum) were proposed as discriminants.

(Gains 7) Late-time acceleration from η-relaxation and unification. Chaps. 8–9 delivered a unified dark sector: cold, stable ρ-structures as dark matter and ultra-long η-relaxation yielding $w_{\rm eff}(t) = -1 + \delta(t)$ with environmental correlations—a testable alternative to rigid Λ.

(Gains 8) Inference and numerics. Chaps. 10–11 provided (i) parameter maps from spectral couplings to observables, (ii) gate-aware priors, (iii) stiff/IMEX integrators, spectral lattices, and E2E validation, and (iv) a synthesis of global gates (Chap. 12) locating the allowed model region.

(Ledger) Primary predictions.

- $\{n_s,\, r,\, f_{\rm NL}^{\rm eff}\}$ with angle-averaged values and an *anisotropy multiplet* $\{g_*,\, \Delta c_s\}$.
- Small δc_T linked to curvature-sector couplings.
- Defect/soliton abundance and σ/M ranges tied to freeze-out scales and core microphysics.
- LSS morphology discriminants exceeding pure-gravity templates.
- Late-time $\delta(z) \equiv 1 + w_{\rm eff}(z)$ and cross-correlations with density/lensing.

16.2 Open Problems and Controlled Risks

Conceptual.

1. **(C1) Origin of the internal symmetry G and vacuum selection.** While Chap. **??** outlined mechanisms for symmetry breaking, a *dynamical* rationale for the *choice* of G (and special cases such as G_2 in Volume I) remains outstanding. *Risk:* overfitting G to phenomenology. *Mitigation:* derive G as an isometry of the spectral fiber endowed with a variational principle minimizing a *spectral curvature functional*; test against anomaly inflow in Vol. III.

2. **(C2) Vacuum alignment and small anisotropy during inflation.** Ensuring $c_s(\theta)$ and c_T anisotropies remain within CMB bounds requires alignment of $\bar{\eta}, \hat{\chi}$. *Risk:* residual quadrupoles. *Mitigation:* quantify alignment potential and friction-induced isotropization; impose priors informed by Chap. 4.

Technical.

1. **(T1) Closure of the spectral RG.** The running of $(\kappa_\eta, M_{\rm eff}, \dots)$ must remain within the hyperbolicity cone. *Risk:* superluminal sectors. *Mitigation:* enforce gate (G2) along the flow; use App. E to propagate modified transfer functions.

2. **(T2) Reheating channels.** Couplings $g_{\eta\Phi}, h_{\eta S}$ must realize $T_{\rm rh}$ high enough for BBN while avoiding excess relics. *Risk:* $\Delta N_{\rm eff}$ tension or late decays. *Mitigation:* joint BBN/CMB posteriors with priors from Chap. 6.

3. **(T3) Defect stability bands.** Core microphysics controls σ/M. *Risk:* SIDM constraints. *Mitigation:* scan core operators on lattices (App. D) and propagate to halos (Chap. 9).

Observational.

1. **(O1) Degeneracies in $(n_s, r, f_{\rm NL})$.** *Risk:* confusion with inflaton models. *Mitigation:* use *joint* anisotropy set $\{g_*, \Delta c_s\}$ and LSS morphology as orthogonal axes.

2. **(O2) Small $\delta(z)$ detectability.** *Risk:* weak leverage on $w_{\rm eff}(z)$. *Mitigation:* cross-correlate with environment (Chap. 8); include GW standard sirens (App. F).

Risk register (summary).

ID	Class	Primary handle	Near-term test
C1	Conceptual	Spectral-curvature variational problem	Anomaly matching (Vol. III)
C2	Conceptual	Alignment potential, Γ_η	CMB g_*, Δc_s
T1	Technical	Gate-preserving RG	Transfer-function fits (App. E)
T2	Technical	$\Gamma_\eta, g_{\eta\Phi}, h_{\eta S}$	BBN/$\Delta N_{\rm eff}$ constraints
T3	Technical	Core operators	σ/M from halos/lensing
O1	Observational	$\{g_*$, morphology$\}$	Joint CMB+LSS pipelines
O2	Observational	Sirens + environment	PTA/LIGO + BAO/SNe

16.3 Interfaces to Future Volumes

(I) Emergent gauge symmetries at the particle level. Volume III will elevate the cosmological isometries of the spectral fiber to *local* gauge dynamics in the particle sector. Key workpackages:

- **I.1 Field content and symmetry breaking:** construct the bundle of internal frames and its connection; identify conditions for $SU(3) \times SU(2) \times U(1)$ emergence; analyze vacuum alignment (Higgs-like), extending the η-wave alignment techniques.
- **I.2 Anomalies and WZW/CS terms:** derive the spectral Wess–Zumino functional; perform 't Hooft anomaly matching between solitonic fermions and emergent gauge currents.
- **I.3 Fermions as solitons:** quantize topological solitons (Vol. I, Ch. 6) with FR constraints; match quantum numbers and derive effective Yukawa textures from spectral distances.

(II) Spectral quantum foundations. A dedicated volume will derive quantum mechanics as the low-energy, linearized dynamics of coherent spectral structures:

- **II.1 Schrödinger as emergent:** obtain the Schrödinger equation from the quadratic expansion around a coherent Φ-soliton; identify the inner product from the quadratic energy form.
- **II.2 Measurement as spectral flooding:** model device/system co-dynamics with $\eta > 0$ channels; compute collapse timescales and pointer states; propose laboratory tests (modified interference with tunable η-like dissipation).
- **II.3 Entanglement as structural non-separability:** quantify nonlocal phase synchronization through χ^μ; design Bell-type tests seeking directional signatures relative to $\hat{\chi}$.

(III) Beyond cosmology: *Applications and stress-tests* in regimes where the medium is structured or strongly anisotropic:

- **III.1 Strong-field and global geometry:** extend the Palatini-like curvature sector to stationary compact objects; reassess horizons, wormhole conditions, and arrow-of-time fragility when $\eta \to 0^+$ locally.
- **III.2 Condensed-matter analogues:** construct laboratory analogs of η-waves and Chladni web formation in metamaterials; validate resonance morphology predictions.
- **III.3 Computational platforms:** release open-source implementations of the integrators (App. D/E), likelihood blocks (App. H), and gate priors (App. G) for community use.

Closing. Volume II demonstrated that a pre-geometric medium with five spectral fields can *generate* a coherent cosmological history and a portfolio of falsifiable signatures, while remaining constrained by a compact set of consistency gates. The next steps—from gauge emergence to quantum foundations—are now *interfaces*, not leaps. They will either tighten the theory into a unified architecture or expose its fault lines. In both outcomes, the program advances: the cosmos is treated not as a stage endowed with laws, but as a *spectral fabric* whose stabilized regularities *are* the laws.